Unit 8

Multiple integrals

Introduction

The previous unit introduced functions of two and more variables, and explained how to differentiate them. In this unit we will integrate such functions.

Suppose that a thin rod lies along the x-axis, with one end at $x = 0$ and the other end at $x = L$. If the rod has a uniform composition, its **linear density** λ (its mass per unit length) is constant everywhere along its length, and the total mass of the rod is $M = \lambda L$.

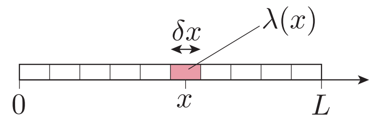


Figure 1 A rod with linear density $\lambda(x)$. The mass contributed by an element centred on x and of length δx is $\lambda(x)\,\delta x$.

A more interesting case arises when the rod is non-uniform (Figure 1). In this case the linear density $\lambda(x)$ is a function of position, and a small element of the rod, centred on x and of length δx, has mass

$$\delta M \simeq \lambda(x)\,\delta x. \tag{1}$$

This equation may be regarded as the definition of the linear density $\lambda(x)$.

Strictly speaking, this is an approximation because we have ignored any variation of $\lambda(x)$ within the element, but the approximation becomes increasingly accurate as the element becomes smaller.

The total mass of the rod can be found be adding together the masses of all its elements. We consider this sum in the limit where the number of elements tends to infinity and the length of each element becomes vanishingly small. In this limit, any approximation involved in equation (1) becomes negligible, and the sum becomes the definite integral

$$M = \int_0^L \lambda(x)\,dx. \tag{2}$$

To find the total mass of the rod, we calculate this integral using the standard rules of calculus. The integral takes a formula for each tiny part and gives an answer for the whole. Not surprisingly, the word *integral* comes from the medieval Latin *integralis* meaning 'forming a whole'.

Crucially for this unit, the tiny elements need not be straight-line segments laid end to end. They could be rectangular elements covering a surface, or tiny brick-shaped elements filling out a volume in three-dimensional space.

Suppose that an oval metal plate of non-uniform composition lies in the xy-plane. In this case each point on the plate can be labelled by its (x, y) coordinates, and the non-uniform composition of the plate can be characterised by a **surface density function** $f(x, y)$, which represents the mass per unit area at any given point (x, y).

The plate can be approximately covered by tiny rectangular area elements aligned with the x- and y-axes, as shown in Figure 2. With rectangular elements the coverage is only approximate, but the approximation is a good one if the elements are small enough.

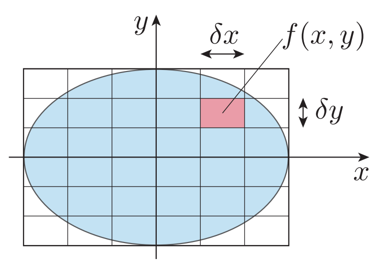


Figure 2 A typical rectangular area element

A typical area element is centred on the point (x, y) and has sides of length δx and δy. This element has area $\delta A = \delta x\,\delta y$, and its mass is

$$\delta M \simeq f(x, y)\,\delta A, \tag{3}$$

where an approximation sign is used because the surface density may vary slightly within the element. However, the approximation becomes increasingly accurate as the element becomes smaller and smaller.

The total mass of the plate is the sum of the masses of all its elements. We consider this sum in the limit where the number of elements tends to infinity and the area of each element becomes vanishingly small. In this limit, the total area occupied by the rectangular elements becomes identical to the area occupied by the plate, and any approximation involved in equation (3) becomes negligible. This is just like the process that led from equation (1) to equation (2), so the sum over all the tiny elements can be regarded as an integral over the area of the plate. The mass of the plate is written as

$$M = \int_S f(x, y)\, dA,$$

where the subscript S on the integral sign indicates that elements exactly cover the surface S of the plate. This expression is called an *area integral*. Of course, we have not told you how to evaluate such an integral, but Section 1 will explain how this is done: by performing *two* definite integrals in succession – one over x and the other over y.

The ideas behind this example can be extended. For instance, instead of integrating over a region in a plane, we can integrate over a curved surface, such as the almost spherical surface of the Earth. If you know the population density (the average number of people per unit area) at each point on Earth, the total human population is found by integrating this population density over the surface of the Earth. Such an integral is called a *surface integral* (rather than an area integral, which is a term that is restricted to planar surfaces).

A cuboid is a rectangular box.

We can also consider three-dimensional objects rather than surfaces. A three-dimensional object can be approximated by many tiny cuboid volume elements aligned with the x-, y- and z-axes. A typical volume element has coordinates (x, y, z) and sides of length δx, δy and δz. Its volume is $\delta V = \delta x\, \delta y\, \delta z$, and its mass is given by

$$\delta M \simeq f(x, y, z)\, \delta V,$$

where $f(x, y, z)$ is the density of the object (its mass per unit volume) at the point (x, y, z). The total mass of the object is the sum of the masses of all the volume elements. We consider this sum in the limit of infinitely many volume elements, each of vanishingly small volume. In this limit, the sum can be expressed as an integral over the volume of the object. We write

$$M = \int_R f(x, y, z)\, dV,$$

where the subscript R shows that the volume elements exactly cover the region R occupied by the object. Such an expression is called a *volume integral*. We have not told you how to evaluate such an integral, but Section 2 will show how this is done: by performing *three* definite integrals in succession – over x, y and z.

The above examples used Cartesian coordinates, but it always possible, and often preferable, to use non-Cartesian coordinates. For example, a given area integral can be evaluated using either Cartesian coordinates or polar coordinates. All coordinate systems give the same answer, but one choice may make life easier than another, and part of the skill of evaluating area, surface and volume integrals is to choose a suitable coordinate system. You will see how to use non-Cartesian coordinates in the second half of this unit.

Uses of surface and volume integrals

Scientists and engineers often need to evaluate area, surface or volume integrals. For example, Figure 3 shows the Hoover Dam in the Colorado River, built in the 1930s to provide irrigation and hydroelectric power. The quantity of concrete used in this structure can be calculated using a volume integral (it is 2.5 million cubic metres). If the curved surface of the dam were to be painted, we could work out the area to be covered by evaluating a surface integral.

In general, volume and surface integrals are useful whenever we have a quantity, such as mass or volume, that is additive. An **additive quantity** is one whose value over a region is equal to the sum of contributions from the region's constituent parts. For example, the mass of any object subdivided into many volume elements is the sum of the masses of these elements. This is what allows us to express the mass of the object first as a sum, and then as an integral.

There are many other physical quantities that are additive, including electric charge, energy and particle number. Each of these quantities can be characterised by a density, which may be per unit area or per unit volume. For example, we may talk about the energy density in the Sun at a given time (Figure 4). The total energy is found by integrating this energy density over the volume of the Sun. We may also talk about the number density of bacteria per unit area on the surface of a laboratory dish. If this number density is modelled by a smooth function of position, then the total number of bacteria on the surface of the dish is found by integrating the number density over the surface of the dish.

Not all physical quantities are additive. For example, we *cannot* say that the temperature of an object is the sum of the temperatures of its parts, and there is no meaningful physical quantity corresponding to the temperature per unit volume. Nevertheless, if temperature is a function $T(x, y, z)$ of position, we may integrate this function over a region, and divide by the volume of the region to give a measure of the *average* temperature in the region.

Figure 3 The Hoover Dam

Some scientists use the term *extensive* instead of additive.

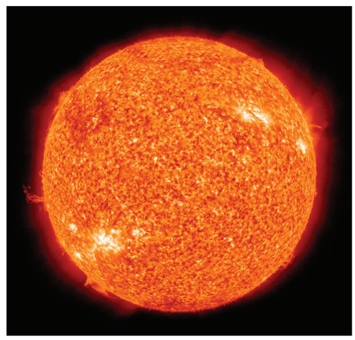


Figure 4 The surface of the Sun imaged by NASA's Solar Dynamics Observatory

Study guide

This unit shows you how to evaluate surface and volume integrals by performing two or three definite integrals in succession. Some of the techniques of integration described in Unit 1 will be used. The unit also uses properties of vector products and determinants that were covered in Unit 4, and the chain rule of partial differentiation, which was introduced in Unit 7.

Section 1 explains how to evaluate area integrals using Cartesian coordinates, and Section 2 deals with volume integrals in Cartesian coordinates.

Section 3 introduces several non-Cartesian coordinate systems and shows how they are used to simplify the evaluation of area and volume integrals. Section 4 gives a review of different types of coordinate system. It unifies all the discussion given earlier in the unit by introducing two important new concepts: *scale factors* and *Jacobian factors*. Finally, Section 5 discusses *surface integrals* over curved surfaces.

1 Area integrals in Cartesian coordinates

In this section, we consider area integrals of functions $f(x, y)$, where x and y are the *Cartesian coordinates* of points in a plane. The regions over which these functions are integrated will also be specified in Cartesian coordinates.

1.1 Area integrals over rectangular regions

We begin with a simple case. Given a function $f(x, y)$ of Cartesian coordinates x and y, we will show how to integrate it over a *rectangular* region in the xy-plane.

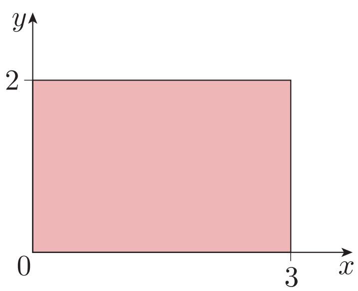


Figure 5 A rectangular slab in the xy-plane

Figure 5 shows a rectangular slab in the xy-plane. This could be a small courtyard, for example. One corner of the slab is at the origin, and the slab extends to $x = 3$ and $y = 2$ (measured in metres). The slab is unevenly covered with a layer of snow whose mass per unit area (measured in kilograms per square metre) at the point (x, y) is

$$f(x, y) = xy^2 \quad \text{for } 0 \le x \le 3 \text{ and } 0 \le y \le 2. \tag{4}$$

The function $f(x, y)$ is defined everywhere on the surface of the slab, and represents the *surface density* of the snow. We can find the total mass of snow on the slab by carrying out a suitable integral.

First, consider the narrow shaded strip in Figure 6, which is centred on $y = y_0$ and has width δy. This strip is made up of many tiny rectangular elements placed end to end, parallel to the x-axis, just as in Figure 1. The mass of a single element in the strip, with linear dimensions δx and δy, centred on the point (x, y_0), is found by multiplying the surface density by the area of the element. From equation (4), the surface density at (x, y_0) is $f(x, y_0) = xy_0^2$, so the mass of the element is

$$\delta M_{\text{element}} \simeq xy_0^2\,\delta x\,\delta y. \tag{5}$$

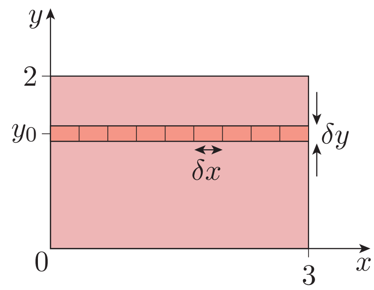


Figure 6 A narrow strip parallel to the x-axis is made up of many tiny elements

An approximation sign is used here because the surface density varies within the element. However, an equals sign will be used from now on, on the understanding that any error will become negligible when we take the limit of vanishingly small elements and integrate.

We can calculate the mass of the strip in Figure 6 by integrating along the x-axis, from one end of the strip to the other. All elements in the strip are centred on $y = y_0$, so this value of y remains constant during the integration. Integrating equation (5) from $x = 0$ to $x = 3$, we get

$$\delta M_{\text{strip}} = \left(\int_{x=0}^{x=3} xy_0^2\,dx\right)\delta y = \left[\tfrac{1}{2}x^2y_0^2\right]_{x=0}^{x=3}\delta y = \tfrac{9}{2}y_0^2\,\delta y.$$

Notice that the integration has been with respect to x, with y held at a constant value $y = y_0$. This is reminiscent of partial differentiation, where we differentiate with respect to x, while holding y constant. In effect, we have *partially integrated* the function $f(x, y)$ with respect to x (although this terminology is rarely used in practice). Notice too that the limits in the integral have been written explicitly as $x = 0$ and $x = 3$, rather than simply as 0 and 3. This is a useful precaution when dealing with a function that depends on more than one variable.

We denoted the constant value of y by y_0 to emphasise the fact that the value of y was held constant during the integration with respect to x. Nevertheless, our formula for the mass of a strip is valid for any value $y = y_0$ within the region of the slab. We can therefore replace y_0 by y to give the mass of snow in *any* narrow strip of width δy, centred on y:

$$\delta M_{\text{strip}} = \tfrac{9}{2}y^2\,\delta y. \tag{6}$$

To find the total mass of snow on the slab, we must add up contributions from all the narrow strips between $y = 0$ and $y = 2$ (see Figure 7). We do this in the limit of vanishingly thin strips, which allows us to replace the sum by a definite integral. Equation (6) gives the mass of a strip, and $9y^2/2$ is the corresponding mass per unit length in the y-direction. To find the total mass of snow on the slab, we integrate $9y^2/2$ with respect to y, from $y = 0$ to $y = 2$. This gives a total mass of

$$M = \int_{y=0}^{y=2} \tfrac{9}{2}y^2\,dy = \tfrac{9}{2}\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=2} = 12,$$

so we conclude that the total mass of snow on the slab is 12 kilograms.

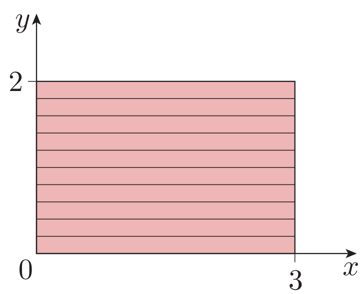


Figure 7 The slab is composed of many narrow strips

Once the principles behind this calculation are understood, it just amounts to doing two integrals in succession, one with respect to x and the other with respect to y. We can write this double integral as

In some texts, the brackets enclosing the inner integral are omitted. This is on the strict understanding that the innermost integral is always evaluated first.

$$M = \int_{y=0}^{y=2} \left(\int_{x=0}^{x=3} xy^2\,dx \right) dy,$$

where the definite integral in brackets is calculated first, treating y as a constant. After the inner integral has been evaluated, and the upper and lower limits for x have been applied, we have a function of y only. This function is integrated with respect to y, and the final answer is obtained by applying the limits for y. This procedure is readily generalised.

Area integral over a rectangular region

The area integral of a function $f(x,y)$ over a region S is denoted by

$$\int_S f(x,y)\,dA.$$

We assume here that $a < b$ and $c < d$.

If S is rectangular and bounded by the lines $x = a$, $x = b$ and $y = c$, $y = d$, the area integral can be obtained as two successive integrals:

$$\int_S f(x,y)\,dA = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\,dx \right) dy. \tag{7}$$

In the integral over x we treat y as a constant.

In equation (7), there is an inner integral (enclosed by brackets) and an outer integral. The inner integral is always performed first. But there is nothing to prevent us from doing things the other way round, integrating first with respect to y and then with respect to x.

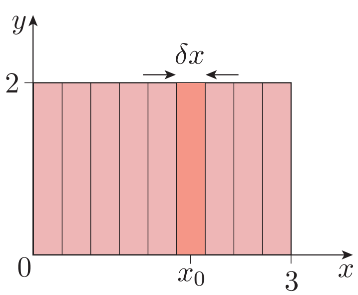


Figure 8 A narrow strip parallel to the y-axis

Returning to the example of the mass of snow on a slab, we could equally well begin by finding the mass of a narrow strip parallel to the y-axis (such as that in Figure 8), and then find the total mass of all such strips.

Integrating the surface density xy^2 first with respect to y and then with respect to x gives

$$\begin{aligned} M &= \int_{x=0}^{x=3} \left(\int_{y=0}^{y=2} xy^2\,dy \right) dx \\ &= \int_{x=0}^{x=3} \left[\tfrac{1}{3}xy^3\right]_{y=0}^{y=2} dx \\ &= \int_{x=0}^{x=3} \tfrac{8}{3}x\,dx = \tfrac{8}{3}\left[\tfrac{1}{2}x^2\right]_{x=0}^{x=3} = \tfrac{8}{3} \times \tfrac{9}{2} = 12, \end{aligned}$$

which is the same as our previous answer. This is just as expected: our decision to subdivide the rectangular slab into thin horizontal strips (as in Figure 7) or into thin vertical strips (as in Figure 8) cannot affect the amount of snow on the slab.

More generally, the area integral in equation (7) can be rewritten as

$$\int_S f(x,y)\,dA = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x,y)\,dy \right) dx. \tag{8}$$

Although the ordering of the integrations makes no difference to the final answer, one ordering may involve easier integrations than the other. You are free to choose whichever order makes the calculation easier.

Example 1

Find the value of the area integral of the function $f(x,y) = y\cos(xy)$ over the rectangle S bounded by the lines $x = 0$, $x = 2$ and $y = \pi/2$, $y = \pi$.

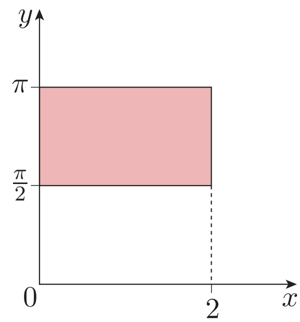


Figure 9 The region of integration for Example 1

Solution

The region of integration is shown in Figure 9. Choosing to integrate over x first, the required area integral is

$$\int_S y\cos(xy)\,dA = \int_{y=\pi/2}^{y=\pi} \left(\int_{x=0}^{x=2} y\cos(xy)\,dx \right) dy.$$

The integral in brackets may look tricky but it is easily evaluated because, when integrating over x, we treat y as a constant. For constant y we have

$$\int_{x=0}^{x=2} y\cos(xy)\,dx = \left[\frac{y\sin(xy)}{y} \right]_{x=0}^{x=2} = \sin(2y).$$

Remember: if a is a constant, $\int a\cos(ax)\,dx = \dfrac{a\sin(ax)}{a} + C.$

So

$$\begin{aligned} \int_S y\cos(xy)\,dA &= \int_{y=\pi/2}^{y=\pi} \sin(2y)\,dy \\ &= \left[-\tfrac{1}{2}\cos(2y) \right]_{y=\pi/2}^{y=\pi} \\ &= -\tfrac{1}{2}\cos(2\pi) + \tfrac{1}{2}\cos(\pi) = -1. \end{aligned}$$

The negative answer is not a problem: it arises because the function $f(x,y) = y\cos(xy)$ is more negative than positive in the given region S.

In the above example, the decision to integrate first with respect to x was key. In theory, we could have integrated first with respect to y, but the integrals would then have been harder. There is no merit is taking a tough route when an easier one lies open, so be prepared to switch the order of integration if your first choice leads to an impasse.

Product functions integrated over rectangular regions

A special case arises when the function to be integrated over a rectangle takes the form $f(x,y) = h(x)\,g(y)$, which is the *product* of a function $h(x)$ of x only and a function $g(y)$ of y only. In this case, the area integral of $h(x)\,g(y)$ over a rectangular region is simply the product of two ordinary integrals:

$$\int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} h(x)\,g(y)\,dx \right) dy = \left(\int_{x=a}^{x=b} h(x)\,dx \right) \times \left(\int_{y=c}^{y=d} g(y)\,dy \right).$$

For example, integrating $f(x,y) = xy^2$ over the region of Figure 5, we have

$$\int_S f(x,y)\,dA = \left(\int_{x=0}^{x=3} x\,dx\right) \times \left(\int_{y=0}^{y=2} y^2\,dy\right)$$
$$= \left[\tfrac{1}{2}x^2\right]_{x=0}^{x=3} \times \left[\tfrac{1}{3}y^3\right]_{y=0}^{y=2} = \tfrac{9}{2} \times \tfrac{8}{3} = 12,$$

in agreement with both our earlier calculations.

However, you must be very careful: splitting an area integral into two factors works only under very special circumstances.

- The integrand $f(x,y)$ must be a product of the form $h(x)\,g(y)$.
- All the limits of integration must be constants: for Cartesian coordinates (x,y), this means that the region of integration must be a rectangle with its edges aligned with the x- and y-axes.

Other area integrals cannot be split in this way. For example, if the integrand is a *sum* of the form $g(x) + h(y)$, the area integral does *not* split into the sum of an integral over x and an integral over y (see Exercise 2).

Exercise 1

Find the value of the area integral of the function $f(x,y) = x^2y^3$ over the square S bounded by the lines $x = 0$, $x = 2$, $y = 1$ and $y = 3$.

Exercise 2

Evaluate the area integral of the function $f(x,y) = 1 + x + y$ over the rectangle S bounded by the lines $x = 1$, $x = 4$, $y = 0$ and $y = 3$.

Exercise 3

Evaluate the area integral

$$I = \int_S \cos(x+y)\,dA,$$

where S is the square region $0 \le x \le \pi$, $0 \le y \le \pi$.

1.2 Area integrals over non-rectangular regions

The area integrals considered so far have all been over rectangular regions aligned with the coordinate axes. This made it easy to determine the limits of integration. For non-rectangular regions we must be more careful in setting up the integrals because the strips are no longer of the same length and the limits on the inner integral depend on the variable in the outer integral. To illustrate this, consider the following example.

Example 2

Find the value of the area integral of the function $f(x,y) = xy$ over the region S bounded by the curves $y = x^2$ and $y = x$ for $0 \le x \le 1$.

In this context, the word 'curve' includes straight lines!

Solution

We begin by drawing a diagram of the region of integration (Figure 10) and choose to integrate first over y and then over x. To determine the limits of the integration over y, consider a vertical strip drawn at an arbitrary fixed value of x within the given region. The ends of the strip lie on the curves $y = x^2$ and $y = x$. So for a given value of x, the lower limit for the y-integration is $y = x^2$, and the upper limit is $y = x$. These limits are the right way round because $x^2 \leq x$ for $0 \leq x \leq 1$, as shown in the diagram.

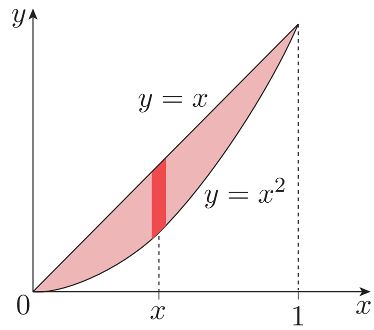


Figure 10 The region of integration for Example 2

The contribution to the area integral from the narrow vertical strip of width δx, centred on x, is found by integrating along the length of the strip:

$$\text{contribution of strip} = \left(\int_{y=x^2}^{y=x} xy\,dy\right)\delta x,$$

where x has a fixed value for a given strip.

A subsequent integration over x sums over all the vertical strips in the region. In this example, the first strip is at $x = 0$ and the last strip is at $x = 1$. Hence the lower and upper limits for the x-integration are $x = 0$ and $x = 1$, and the complete area integral is

$$\int_S xy\,dA = \int_{x=0}^{x=1}\left(\int_{y=x^2}^{y=x} xy\,dy\right)dx. \tag{9}$$

The integral enclosed by brackets is carried out first. This is with respect to y and is evaluated by treating x as a constant, giving

$$\int_{y=x^2}^{y=x} xy\,dy = \left[\tfrac{1}{2}xy^2\right]_{y=x^2}^{y=x} = \tfrac{1}{2}(x^3 - x^5).$$

The result is a function of x only. This is finally integrated over x to give the area integral:

$$\int_S xy\,dA = \int_{x=0}^{x=1} \tfrac{1}{2}(x^3 - x^5)\,dx = \tfrac{1}{2}\left[\tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_{x=0}^{x=1} = \tfrac{1}{24}.$$

Let us review this example. We started by drawing a diagram that helped us to find the limits of integration. This is an essential step. We chose to integrate first with respect to y, with x held constant. The limits of the inner y-integral were functions of x, and the limits of the outer x-integral were constants. The area integral was then found by two successive integrations, the first over y (with x held constant) and the second over x.

Although the integrand in this case is a product of a function of x and a function of y, the area integral does *not* reduce to the product of two ordinary integrals. This is because the limits of integration are not all constants. In equation (9), the limits of the y-integral depend on x, so we must do this integral first, and then integrate the result over x.

The general method is readily extended to other area integrals.

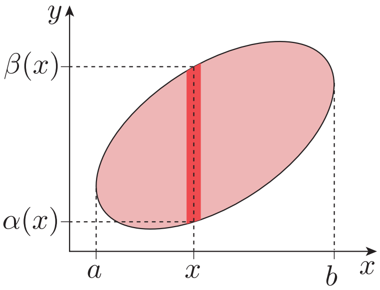


Figure 11 The region of integration for an area integral. The y-limits are given by the equations $y = \alpha(x)$ and $y = \beta(x)$ of the boundary curves.

Procedure 1 Evaluating an area integral

To evaluate an area integral $\int_S f(x, y)\, dA$ over any given region S of the xy-plane, we must decide which integral to do first – that over x or that over y. The following steps assume that we have chosen to integrate first with respect to y, and then with respect to x.

1. Draw a diagram showing the region of integration S.
2. Draw a vertical strip parallel to the y-axis, centred on x, and spanning the region (as in Figure 11). Determine the lower limit $y = \alpha(x)$ and the upper limit $y = \beta(x)$ for this strip. These are the limits for the y-integration (the inner integration). For non-rectangular regions, they are non-constant functions of x.
3. Determine the minimum value $x = a$ and the maximum value $x = b$ for x-values throughout the region. These are the limits for the x-integration (the outer integration), and are always constants.
4. Write down the area integral as
$$\int_S f(x, y)\, dA = \int_{x=a}^{x=b} \left(\int_{y=\alpha(x)}^{y=\beta(x)} f(x, y)\, dy \right) dx. \tag{10}$$
5. Evaluate the inner integral over y first, holding x constant, and substituting in the limits of integration. This gives a function
$$g(x) = \int_{y=\alpha(x)}^{y=\beta(x)} f(x, y)\, dy,$$
which remains to be integrated over x.
6. Evaluate the remaining definite integral of $g(x)$ over x.

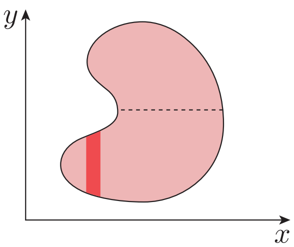


Figure 12 A region for which Procedure 1 fails. The problem is overcome by dividing the region into parts, as shown by the dashed line.

Step 2 of this procedure may sometimes fail. In Figure 12, for example, a strip parallel to the y-axis reaches the boundary of the region before it has spanned the whole region. Such cases are dealt with by breaking the region into smaller parts, but you will not meet this complication in this module.

Exercise 4

Evaluate the area integral of the function $f(x, y) = x - y$ over the triangular region S bounded by the lines $y = x - 1$, $x = 3$ and $y = 0$.

The simplest application of an area integral is to find the area of a given region S in the xy-plane. The area is given by integrating the constant function $f(x, y) = 1$ over the region.

$$\text{Area of region } S = \int_S 1\, dA. \tag{11}$$

Exercise 5

Find the area of the region between the curve $y = \cos x$ and the straight line $y = 1 - 2x/\pi$, for $0 \leq x \leq \pi/2$.

Note that $\cos x \geq 1 - 2x/\pi$ for $0 \leq x \leq \pi/2$.

In Procedure 1 we chose to organise the area into vertical strips and integrate first with respect to y. We can also organise the area into horizontal strips and integrate first with respect to x. In effect, we would then continue to use Procedure 1 but with x and y interchanged. In this alternative ordering the area integral takes the form

$$\int_S f(x,y)\,dA = \int_{y=a}^{y=b} \left(\int_{x=u(y)}^{x=v(y)} f(x,y)\,dx \right) dy. \tag{12}$$

Here, the inner integration is over x with limits of integration that are functions of y, and the outer integration is over y with limits of integration that are constants. The inner integration is carried out first; it yields a function of y, which is then integrated to produce the final answer (which does not depend on x or y). For well-behaved functions, the final answer does not depend on whether we divide the area into vertical or horizontal strips, so equation (12) gives the same answer as equation (10); the choice of method is just one of convenience.

In the special case of an area integral over a rectangular region aligned with the coordinate axes, the limits of integration are all constants. But usually, the limits of the inner integral are not constants. This has an important consequence for calculations.

Rethinking the limits of integration

If we choose to do the x-integral first, as in equation (12), then the limits of integration must be rethought from scratch. This is done by drawing a new sketch of the region, with horizontal strips running parallel to the x-axis, rather than vertical strips parallel to the y-axis.

Let us return to the area integral of $f(x,y) = x - y$ over the triangle S bounded by the lines $y = x - 1$, $x = 3$ and $y = 0$. This was evaluated in Exercise 4 by integrating first over y and then over x. Now we will evaluate the same integral, but with the integrals performed in the opposite order.

A new sketch is needed, and this is given by Figure 13, which shows a typical horizontal strip across the region of integration. This stretches from $x = y + 1$ to $x = 3$. The minimum and maximum values of y are $y = 0$ and $y = 2$, and these are the lower and upper limits of the y-integration. So the area integral is expressed as

$$\int_S f(x,y)\,dA = \int_{y=0}^{y=2} \left(\int_{x=y+1}^{x=3} (x - y)\,dx \right) dy.$$

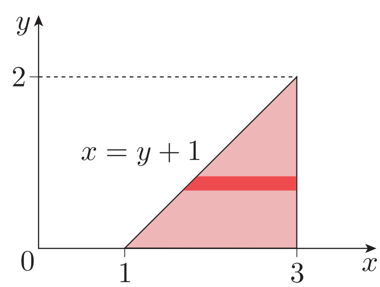


Figure 13

This area integral looks different to that in Exercise 4, but it leads to the same answer. Integrating over x while treating y as a constant gives

$$\begin{aligned}\int_S f(x,y)\,dA &= \int_{y=0}^{y=2} \left[\tfrac{1}{2}x^2 - xy\right]_{x=y+1}^{x=3} dy \\ &= \int_{y=0}^{y=2} \left(\tfrac{9}{2} - 3y - \tfrac{1}{2}(y+1)^2 + y(y+1)\right) dy \\ &= \int_{y=0}^{y=2} \left(4 - 3y + \tfrac{1}{2}y^2\right) dy.\end{aligned}$$

The final integral over y then gives

$$\begin{aligned}\int_S f(x,y)\,dA &= \left[4y - \tfrac{3}{2}y^2 + \tfrac{1}{6}y^3\right]_{y=0}^{y=2} \\ &= 8 - 6 + \tfrac{4}{3} = \tfrac{10}{3},\end{aligned}$$

as before.

Exercise 6

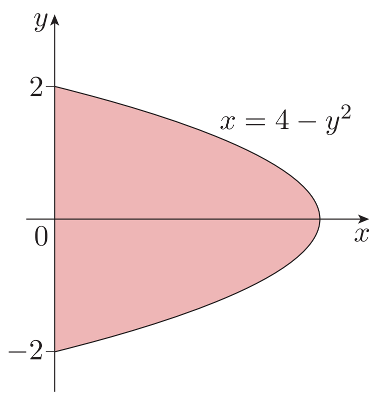
Sketch the areas of integration for each of the following area integrals, and write down alternative expressions for the same integrals, but with the order of integration reversed.

(a) $\displaystyle\int_{x=0}^{x=2} \left(\int_{y=x/2}^{y=1} f(x,y)\,dy\right) dx$

(b) $\displaystyle\int_{x=0}^{x=2} \left(\int_{y=0}^{y=x/2} f(x,y)\,dy\right) dx$

Exercise 7

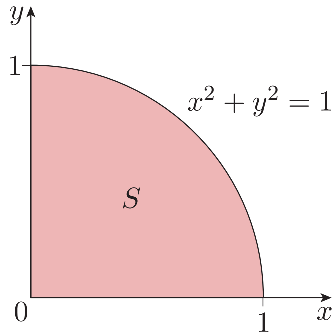
The shaded region in the figure below is bounded by the y-axis and the curve $x = 4 - y^2$.



Find the area of this region by integrating first over x and then over y.

Exercise 8

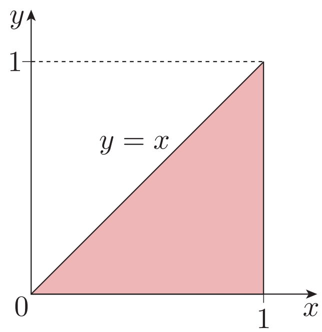
Evaluate the area integral of the function $f(x, y) = x$ over the shaded region S in the figure below, which is a quarter-disc $x^2 + y^2 \leq 1$ with $x \geq 0$ and $y \geq 0$.



(*Hint*: In this case, it is easier to integrate over x first, and then over y.)

Exercise 9

Evaluate the area integral of $f(x, y) = \exp(x^2)$ over the triangular region in the figure below, which is enclosed by the lines $x = 1$, $y = 0$ and $y = x$.



(*Hint*: The integrals over x and y can be written down in either order, but only one of these orderings gives integrals that can be done!)

2 Volume integrals in Cartesian coordinates

We now consider volume integrals. This section uses Cartesian coordinates to evaluate volume integrals over cuboid and some non-cuboid regions. The approach is similar to that used for area integrals, but there are now three coordinates, x, y and z, to consider, and three integrals to perform in succession.

2.1 Volume integrals over cuboid regions

Suppose that we want to find the total mass of an object from its density. Within the object, the density is described by a function $f(x, y, z)$ of Cartesian coordinates (x, y, z). This function allows us to find the mass of a small volume element. An element of volume δV, centred on the point (x, y, z), has mass

$$\delta M = f(x, y, z)\,\delta V.$$

Strictly speaking, this is an approximation because the density function may vary slightly within the volume element, but if the volume element is small enough, any such variation is negligible.

Now imagine subdividing the object into a vast number of tiny non-overlapping volume elements. The total mass of the object is the sum of the masses of its elements. Rather than adding up an immense number of small masses, we take the limit of the sum as the volume of each element tends to zero, and the number of elements tends to infinity. In this limit, the sum becomes an integral and the approximations become exact. The mass of the object can then be written as a volume integral:

$$M = \int_R f(x, y, z)\,dV,$$

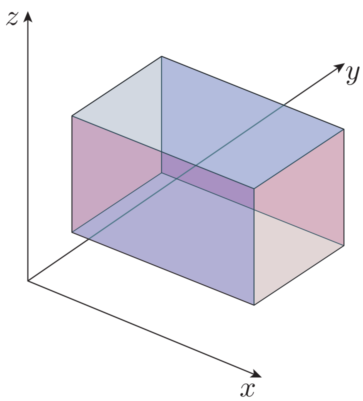


Figure 14 A cuboid

where R is the region occupied by the object, and $f(x, y, z)$ is the density function.

In this subsection, we explain how volume integrals are calculated in the special case where the region of integration is a **cuboid** (i.e. a rectangular block). We choose a Cartesian coordinate system with axes aligned with the faces of the block (Figure 14). These faces lie in the coordinate planes $x = a_1$, $x = a_2$, $y = b_1$, $y = b_2$, $z = c_1$ and $z = c_2$, where $a_1, a_2, \ldots, c_2$ are constants. (Here we use subscripts to distinguish the six constants that define the cuboid. This is neater than using six different letters of the alphabet.)

Suppose that we want to integrate the density function $f(x, y, z)$ over this cuboid. We start with a tiny volume element, which is a tiny block with linear dimensions δx, δy and δz, centred on the point (x, y, z). The volume of this element is $\delta V = \delta x\,\delta y\,\delta z$, and its mass is

$$\text{mass of element} = f(x, y, z)\,\delta x\,\delta y\,\delta z.$$

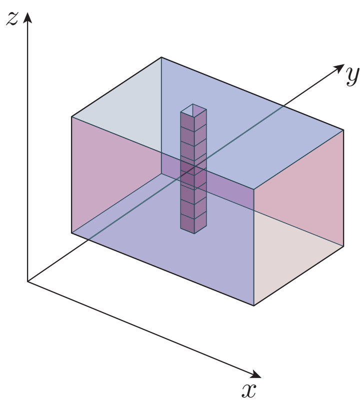


Figure 15 A narrow column within a cuboid

Figure 15 shows how similar tiny elements can be stacked on top of one another to produce a vertical column extending from the bottom to the top face of the cuboid. All the tiny blocks are centred on the same values of x and y, but the value of z varies from $z = c_1$ on the bottom face to $z = c_2$ on the top face. The mass of the column is the sum of the masses of the volume elements that it contains, and in the limit of very small volume elements this can be expressed as an integral:

$$\text{mass of column} = \left(\int_{z=c_1}^{z=c_2} f(x, y, z)\,dz\right)\delta x\,\delta y,$$

where x and y are held constant during the integration over z. The mass of the column remains a function of x and y because the density function may vary from point to point.

Next, we stick many columns together to form a slice running across the cuboid at constant x (Figure 16). The mass of the slice is the sum of the masses of the columns that it contains, and in the limit of very narrow columns this can be expressed as the integral

$$\text{mass of slice} = \left(\int_{y=b_1}^{y=b_2} \left(\int_{z=c_1}^{z=c_2} f(x,y,z)\, dz \right) dy \right) \delta x,$$

where x is held constant during the integration over y. The mass of each slice is a function of x only.

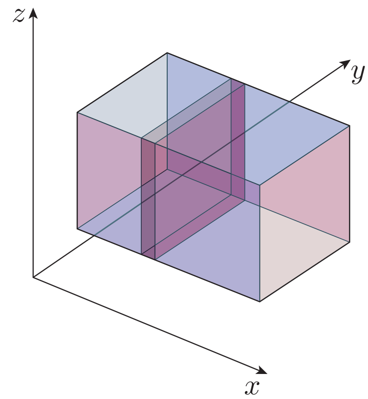


Figure 16 A thin slice within a cuboid

Finally, we join all the slices together to form the complete cuboid. The mass of the cuboid is the sum of the masses of the slices that it contains, and in the limit of very thin slices this can be expressed as

$$\text{mass of cuboid} = \int_{x=a_1}^{x=a_2} \left(\int_{y=b_1}^{y=b_2} \left(\int_{z=c_1}^{z=c_2} f(x,y,z)\, dz \right) dy \right) dx. \qquad (13)$$

This is a typical volume integral over a cuboid. There is no need to go through the steps of dividing the cuboid into blocks, columns and slices every time. You can go straight to the conclusion given by equation (13). More generally, the volume integral of any function $f(x,y,z)$ over a cuboid is given by a similar expression.

Volume integral over a cuboid

Given a function $f(x,y,z)$ of Cartesian coordinates x, y and z, and a cuboid region R, with faces lying in the coordinate planes $x = a_1$, $x = a_2$, $y = b_1$, $y = b_2$, $z = c_1$ and $z = c_2$, the volume integral of f over R is given by

$$\int_R f(x,y,z)\, dV = \int_{x=a_1}^{x=a_2} \left(\int_{y=b_1}^{y=b_2} \left(\int_{z=c_1}^{z=c_2} f(x,y,z)\, dz \right) dy \right) dx. \qquad (14)$$

We assume here that $a_1 < a_2$, $b_1 < b_2$ and $c_1 < c_2$.

If $f(x,y,z)$ represents density (the mass per unit volume), then the volume integral in equation (14) gives the total mass contained in the region R. If $f(x,y,z) = 1$, then the integral gives the volume of the region R.

In all cases, this volume integral is evaluated from the inside out. First, we integrate over z, holding x and y constant. Then we integrate over y, holding x constant. Finally, we integrate over x. None of this should surprise you. It follows exactly the same pattern as for area integrals over rectangular regions, but we must now integrate over the three coordinates x, y *and* z, rather than just two.

As for area integrals over rectangular regions, the order of integration makes no difference to the final answer. For example, the above integral can also be written as

$$\int_R f(x,y,z)\,dV = \int_{z=c_1}^{z=c_2}\left(\int_{y=b_1}^{y=b_2}\left(\int_{x=a_1}^{x=a_2} f(x,y,z)\,dx\right)dy\right)dz.$$

Although the final result is the same, the effort needed for the integration may depend on the choice made.

For area integrals, you saw that product functions integrated over rectangular regions can be expressed as the product of two ordinary integrals. A similar result applies to volume integrals. The volume integral of a product function $f(x,y,z) = u(x)\,v(y)\,w(z)$ over a *cuboid* region is simply the product of three ordinary integrals:

$$\int_R f(x,y,z)\,dV = \int_{z=c_1}^{z=c_2} w(z)\,dz \times \int_{y=b_1}^{y=b_2} v(y)\,dy \times \int_{x=a_1}^{x=a_2} u(x)\,dx.$$

But you must be very careful: splitting a volume integral into three factors works only under very special circumstances.

- The integrand $f(x,y,z)$ must be a product of the form $u(x)\,v(y)\,w(z)$.
- All the limits of integration must be constants: for Cartesian coordinates (x,y,z), this means that the region of integration must be a cuboid with its faces aligned with the x-, y- and z-axes.

Other volume integrals cannot be split in this way.

Example 3

A cube has faces at $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$, where lengths are measured in metres. The non-uniform density of the cube (in kilograms per cubic metre) is described by $f(x,y,z) = x^2+y^2+z^2$. Determine the mass of the cube.

Solution

The mass of the cube is given by the volume integral

$$M = \int_R (x^2+y^2+z^2)\,dV,$$

where R is the volume occupied by the cube. Inserting the appropriate limits, we have

$$M = \int_{x=0}^{x=1}\left(\int_{y=0}^{y=1}\left(\int_{z=0}^{z=1}(x^2+y^2+z^2)\,dz\right)dy\right)dx.$$

The inner integral is evaluated first. This is an integral over z with x and y treated as constants. We get

$$\begin{aligned} M &= \int_{x=0}^{x=1}\left(\int_{y=0}^{y=1}\left[x^2z+y^2z+\tfrac{1}{3}z^3\right]_{z=0}^{z=1}dy\right)dx \\ &= \int_{x=0}^{x=1}\left(\int_{y=0}^{y=1}\left(x^2+y^2+\tfrac{1}{3}\right)dy\right)dx. \end{aligned}$$

Next, we integrate over y with x treated as a constant. We get

$$\begin{aligned} M &= \int_{x=0}^{x=1} \left[x^2 y + \tfrac{1}{3}y^3 + \tfrac{1}{3}y\right]_{y=0}^{y=1} dx \\ &= \int_{x=0}^{x=1} \left(x^2 + \tfrac{2}{3}\right) dx. \end{aligned}$$

Finally, we integrate over x to obtain

$$M = \left[\tfrac{1}{3}x^3 + \tfrac{2}{3}x\right]_{x=0}^{x=1} = 1.$$

So the mass of the cube is 1 kilogram.

Exercise 10

A cuboid block R has faces at $x = 0$, $x = 2$, $y = 1$, $y = 2$, $z = 2$ and $z = 5$, where lengths are measured in metres. The non-uniform density of the block (in kilograms per cubic metre) is given by $f(x, y, z) = x + y + z$. Find the mass of the block.

Exercise 11

Show that the function $f(x, y, z) = xyz\, e^{-(x^2+y^2+z^2)}$ can be expressed as a product of the form $u(x)\, v(y)\, w(z)$. Hence evaluate the volume integral

$$I = \int_R xyz\, e^{-(x^2+y^2+z^2)}\, dV,$$

where R is a cube with faces at $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$.

2.2 Volume integrals over non-cuboid regions

A cuboid region of integration is, of course, an exceptional case. For shapes other than cuboids, Cartesian coordinates can still be used, but care is needed with the limits of integration.

Remember what happened for area integrals over non-rectangular regions – the limits of the inner integral depended on the variable of integration in the outer integral. Something very similar happens for volume integrals over non-cuboid regions.

Volume integral over a non-cuboid region

The volume integral of a function $f(x, y, z)$ over a non-cuboid region R can be written in the form

$$\int_R f(x, y, z)\, dV = \int_{x=a}^{x=b} \left(\int_{y=\alpha(x)}^{y=\beta(x)} \left(\int_{z=u(x,y)}^{z=v(x,y)} f(x, y, z)\, dz \right) dy \right) dx. \tag{15}$$

This involves three integrations performed successively, from the innermost to the outermost. In this case, we integrate first with respect to z (holding x and y constant). This gives us a function of x and y, which is integrated with respect to y (holding x constant). This finally gives a function of x, which is integrated over x.

The limits for the innermost integral depend on the variables of integration x and y in the outer two integrals. The limits for the middle integral depend on the variable of integration x in the outermost integral, and the limits of the outermost integral are constants.

In the special case where $f(x, y, z) = 1$, equation (15) gives the volume of the region.

$$\text{Volume of region} = \int_{x=a}^{x=b} \left(\int_{y=\alpha(x)}^{y=\beta(x)} \left(\int_{z=u(x,y)}^{z=v(x,y)} 1\, dz \right) dy \right) dx. \quad (16)$$

Sometimes the trickiest part of the calculation is finding the limits of integration that define the region R. This is best done by sketching *two* diagrams, as illustrated in the following example.

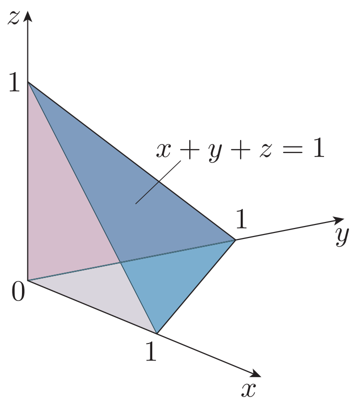


Figure 17 The region of integration for Example 4

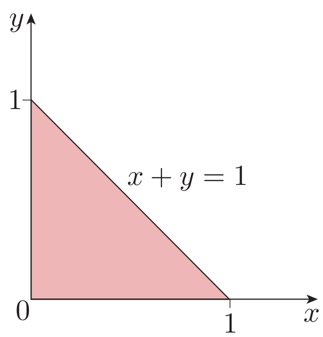


Figure 18 Projection of the region of integration onto the xy-plane

Example 4

Find the volume integral of the function $f(x, y, z) = z^2$ over a pyramid R whose faces are given by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. This pyramid has vertices at points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

Solution

The pyramid is sketched in Figure 17. The plane $x + y + z = 1$ meets the xy-plane in the line $x + y = 1$, so the triangular base of the pyramid is as shown in Figure 18.

The limits of integration can now be determined from our sketches. Above any tiny rectangular area element in the base of the pyramid, we can imagine a narrow column extending up to the blue shaded plane in Figure 17. By summing over all such columns, we cover the whole region of integration inside the pyramid. For a single column, rising from a point (x, y), we integrate over z from $z = 0$ to $z = 1 - x - y$. From Figure 18, we see that a slice at constant x can be produced by integrating over y from $y = 0$ to $y = 1 - x$. And to include all the slices, we must integrate over x from $x = 0$ to $x = 1$.

The order of integration can be checked from the nature of the limits. The z-integral *must* be placed innermost because its limits depend on both x and y. The y-integral comes next because its limits depend on x. Finally, the x-integral is outermost because its limits are constants. The function to be integrated is $f(x, y, z) = z^2$, so the required volume integral is

$$\int_R f(x, y, z)\, dV = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x} \left(\int_{z=0}^{z=1-x-y} z^2\, dz \right) dy \right) dx.$$

The hardest part of the problem is over; now we just need to do the three integrations. We begin by integrating over z, holding both x and y constant:

$$\int_{z=0}^{z=1-x-y} z^2\,dz = \left[\tfrac{1}{3}z^3\right]_{z=0}^{z=1-x-y} = \tfrac{1}{3}(1-x-y)^3.$$

We then integrate this function over y, holding x constant. You can see by inspection that when k is a constant,

$$\int (k-y)^3\,dy = -\tfrac{1}{4}(k-y)^4 + \text{constant},$$

This integral can be checked by differentiation. Alternatively, you could use the change of variable $u = k - y$.

so, replacing k by $1-x$, we get

$$\int_{y=0}^{y=1-x} \tfrac{1}{3}(1-x-y)^3\,dy = \left[-\tfrac{1}{12}(1-x-y)^4\right]_{y=0}^{y=1-x} = \tfrac{1}{12}(1-x)^4.$$

Finally, we integrate over x to obtain

$$\int_R f(x,y,z)\,dV = \int_{x=0}^{x=1} \tfrac{1}{12}(1-x)^4\,dx = \left[\tfrac{1}{12}\left(-\tfrac{1}{5}(1-x)^5\right)\right]_{x=0}^{x=1} = \tfrac{1}{60}.$$

The same volume integral can be written in a variety of ways, depending on how we order the integrals. For example, we could have projected the region onto the yz-plane, and chosen to integrate first over x, then over y, and finally over z. The volume integral in Example 4 would then be written as

$$\int_R f(x,y,z)\,dV = \int_{z=0}^{z=1}\left(\int_{y=0}^{y=1-z}\left(\int_{x=0}^{x=1-y-z} z^2\,dx\right)dy\right)dz. \qquad (17)$$

This gives the same answer as the volume integral in Example 4, as you can now check.

Exercise 12

Verify that equation (17) gives the same answer as that in Example 4.

Finding the limits of integration is a key step in all problems of this kind, and requires great care. Let us review how this is done. It is usually helpful to draw *two* diagrams to visualise the geometry – a perspective view of the three-dimensional region of integration and a plan view showing the projection of the region onto a coordinate plane (the xy-plane in Example 4).

Focusing on a tiny element with coordinates (x, y) in the projection onto the xy-plane, and extending upwards in a column within the region, we obtain limits for the inner integration over z. In general, these limits depend on x and y. Then we imagine sticking many columns together, producing a slice across the region at constant x. The y-values at the extremities of a typical slice are evident in the sketch showing the projection of the region onto the xy-plane. These are the limits for the middle integral over y, which in general depend on x. Finally, the

minimum and maximum values of x in the projection onto the xy-plane give the constant limits for the outer integral over x.

When you nest the three integrals to form a triple integral, it is worth checking that the following rule is obeyed.

Rule for the ordering of integrals in a triple integral

The limits of integration of a given integral can depend only on the variables of integration of integrals that lie further *outside* it (and are done after it). The limits of the outer integral are always constants.

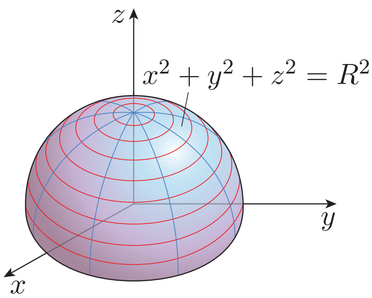


Figure 19 A hemisphere

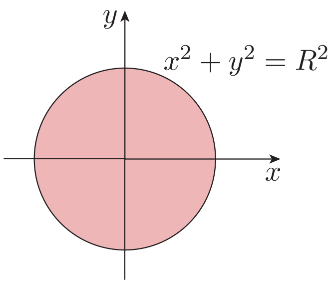


Figure 20 Projection of the hemisphere onto the xy-plane

Example 5

Figure 19 shows a hemisphere of radius R, with its base in the xy-plane, centred on the origin. In Cartesian coordinates, points on the curved surface of the hemisphere have $x^2+y^2+z^2=R^2$. Write down an integral expression that gives the volume of this hemisphere. Do not spend any time evaluating the integrals.

Solution

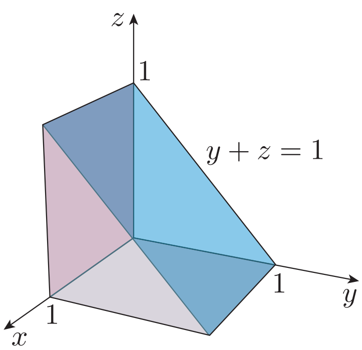
Consider Figure 19 and the given equation for the curved surface of the hemisphere. For a given element with coordinates (x, y), the limits of the inner integration are $z=0$ and $z=\sqrt{R^2-x^2-y^2}$.

We draw a two-dimensional view of the projection of the hemisphere onto the xy-plane. This is the disc of radius R shown in Figure 20. Choosing to integrate next over y, we see that for a strip centred on x, the limits are $y=-\sqrt{R^2-x^2}$ and $y=+\sqrt{R^2-x^2}$. The limits for the final x-integration are $x=-R$ and $x=+R$.

To find the volume of the hemisphere, we integrate the function $f(x,y,z)=1$ over the hemispherical region. So the volume is given by

$$V=\int_{x=-R}^{x=R}\left(\int_{y=-\sqrt{R^2-x^2}}^{y=\sqrt{R^2-x^2}}\left(\int_{z=0}^{z=\sqrt{R^2-x^2-y^2}}1\,dz\right)dy\right)dx.$$

This illustrates the process of finding suitable limits, but the integrals are lengthy and will not be done here. Later in this unit you will see that there are better methods to use in this case, based on *non-Cartesian coordinates*.



Exercise 13

Find the value of the volume integral of the function $f(x,y,z)=x^2yz$ over the wedge-shaped region shown in the margin, which is bounded by the planes $z=0$, $y=0$, $x=0$, $x=1$ and $y+z=1$.

Sometimes the mathematical statement of a problem already specifies the limits of integration, and we can skip the stage of drawing diagrams, as in the following exercise.

Exercise 14

A region in three-dimensional space is defined by

$$0 \le x \le y^2 + z^2, \quad 0 \le y \le z, \quad 0 \le z \le 1,$$

where lengths are measured in metres. What is the volume of this region?

3 Using non-Cartesian coordinates

In principle, it is possible to use the methods of Sections 1 and 2 to evaluate any area or volume integral in Cartesian coordinates x, y and z – possible, but not always wise! For simple shapes such as discs, cylinders and spheres, it is generally much easier to use a different approach based on *non-Cartesian coordinates*. This section gives an introduction to the three most commonly used non-Cartesian systems: *polar coordinates*, *cylindrical coordinates* and *spherical coordinates*.

3.1 Area integrals in polar coordinates

Polar coordinates

Points in a plane are often specified by Cartesian coordinates (x, y), but this is not essential. An alternative choice is to use **polar coordinates** (r, ϕ), as shown in Figure 21.

Sometimes, the polar coordinate is denoted by θ rather than ϕ. Our present choice is deliberate and will have advantages when we compare polar, cylindrical and spherical coordinates.

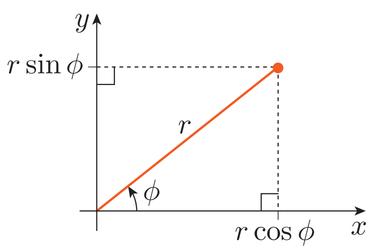


Figure 21 Polar coordinates

- The **radial coordinate** r is the distance of the point from the origin and lies in the range $0 \le r < \infty$.
- The **angular coordinate** ϕ is the angle measured anticlockwise from the positive x-direction, measured in radians. The value of ϕ is not unique because we can add any integer multiple of 2π radians to it, and still be describing the same point. We often take ϕ to lie in the range $0 \le \phi < 2\pi$, but other choices (such as $-\pi \le \phi < \pi$) are equally valid.

Using trigonometry in Figure 21, we see that Cartesian and polar coordinates are related as follows.

$$x = r\cos\phi, \quad y = r\sin\phi. \tag{18}$$

So if we know the polar coordinates (r, ϕ) of a point, we can easily find its Cartesian coordinates (x, y).

Area integrals in polar coordinates

When an area integral is set up in a given coordinate system, a key step is to subdivide the region of integration into a set of tiny area elements.

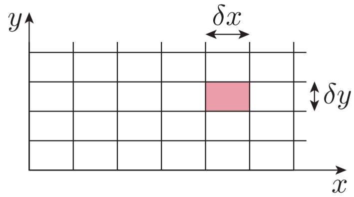


Figure 22 A Cartesian grid

In Cartesian coordinates, this is achieved by drawing lines parallel to the coordinate axes (Figure 22). Along each horizontal line, x varies at constant y. Along each vertical line, y varies at constant x. These lines produce a rectangular grid that divides the xy-plane into tiny rectangular area elements, such as that shaded in Figure 22. This element has area

$$\delta A = \delta x\,\delta y.$$

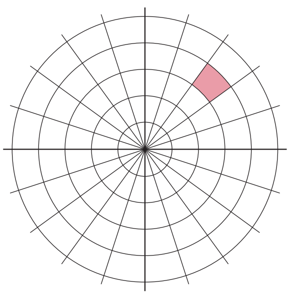


Figure 23 A polar grid

Something similar is done for polar coordinates. As shown in Figure 23, we create a grid from a set of radial lines (spokes) and a set of circles. Each spoke is a line along which r increases at constant ϕ. Each circle is a curve along which ϕ varies at constant r. Taken together, the spokes and circles divide the xy-plane into tiny area elements, such as that shaded in Figure 23.

Figure 24 shows a tiny area element, with its size exaggerated for clarity. The element is bounded by radial spokes with $\phi = \phi_0$ and $\phi = \phi_0 + \delta\phi$, and circular arcs with $r = r_0$ and $r = r_0 + \delta r$. The spokes and circular arcs meet at right angles to one another, so if the element is *extremely* small, it can be approximated by a rectangle. The sides running along the spokes have length δr. Because the angle ϕ is measured in radians, the sides running round the circular arcs have length $r_0\,\delta\phi$. So the element has area

Recall the formula $r\,\delta\phi$ for the length of an arc subtended on a circle of radius r by the angle $\delta\phi$, where $\delta\phi$ is measured in radians.

$$\delta A \simeq \delta r \times r_0\,\delta\phi = r_0\,\delta r\,\delta\phi.$$

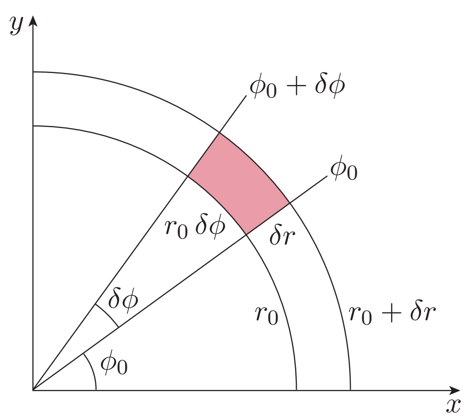


Figure 24 An area element in polar coordinates (enlarged for clarity)

Of course, there is nothing special about the values $r = r_0$ and $\phi = \phi_0$, so we can drop the subscripts and say that an area element centred on (r, ϕ) has area

$$\delta A \simeq r\,\delta r\,\delta\phi, \tag{19}$$

and if we are interested in the area integral of a function $f(r, \phi)$ over some region of the plane, the contribution made by the area element is

$$f(r,\phi)\,\delta A \simeq f(r,\phi)\,r\,\delta r\,\delta\phi.$$

The area integral of $f(r, \phi)$ over a given region in the plane is obtained by adding contributions from all the area elements that make up the region. We do this in the limit of an infinite number of infinitesimally small elements. Then our approximations become exact, and at the same time the sum becomes an integral.

As a definite case, let us take the region of integration to be a disc of radius R, centred on the origin. Then the limits for the r-integral are $r = 0$ and $r = R$, and the limits for the ϕ-integral may be taken to be $\phi = 0$ and $\phi = 2\pi$. The area integral can therefore be written as follows.

Area integral over a disc in polar coordinates

The area integral of $f(r, \phi)$ over a disc S of radius R centred on the origin is

$$\int_S f(r,\phi)\,dA = \int_{\phi=0}^{\phi=2\pi} \left(\int_{r=0}^{r=R} f(r,\phi)\,r\,dr \right) d\phi, \tag{20}$$

where we have chosen to integrate first over r, and then over ϕ.

The reverse order is also valid: with the inner integral over ϕ, and the outer integral over r.

Take careful note of the factor r that appears in equation (20) – and never make the mistake of leaving it out! It occurs in all area integrals based on polar coordinates. There are two ways of seeing why this factor must be included:

- Figure 23 shows that the area elements grow in size as we move away from the origin. If the area element were simply $\delta r\,\delta\phi$, this fact would not be respected.
- The expression $\delta r\,\delta\phi$ has the dimensions of length (because δr is a length and $\delta\phi$ is dimensionless). This is not suitable for an area element; the extra factor r ensures that the area element has the required dimensions of length squared.

The significance of the area integral in equation (20) is similar to that of an area integral in Cartesian coordinates. For example, if $f(r, \phi)$ is the surface density (the mass per unit area) at a point on the disc with polar coordinates (r, ϕ), then equation (20) gives the total mass of the disc.

If $f(r,\phi)=1$, then equation (20) gives the area of the disc. This evaluates to

$$\begin{aligned}\int_S 1\,dA &= \int_{\phi=0}^{\phi=2\pi}\left(\int_{r=0}^{r=R} r\,dr\right)d\phi\\ &= \int_{\phi=0}^{\phi=2\pi}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=R}\,d\phi\\ &= \int_{\phi=0}^{\phi=2\pi}\tfrac{1}{2}R^2\,d\phi = \pi R^2,\end{aligned}$$

as you would expect. The calculation is much easier in polar coordinates than in Cartesian coordinates. This is because when we integrate over a disc in polar coordinates, the limits of integration are constants that are easily determined.

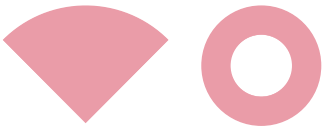


Figure 25 Some regions of integration suitable for polar coordinates

A similar simplification occurs for the regions of integration in Figure 25, which also have constant limits of integration in polar coordinates. Polar coordinates are usually the preferred choice for such regions. This is true even if the integrand is initially specified in Cartesian coordinates because it can be easily converted to polar coordinates using equations (18).

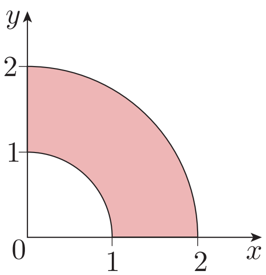


Figure 26 The annular region of integration in Example 6

Example 6

Evaluate the area integral of the function $f(x,y)=xy^2$ over the annular sector S shown in Figure 26. In polar coordinates, S is defined by $1\le r\le 2$ and $0\le\phi\le\pi/2$.

Solution

The shape of the region of integration suggests the use of polar coordinates. Using equations (18) we have

$$xy^2=(r\cos\phi)\times(r\sin\phi)^2=r^3\sin^2\phi\cos\phi,$$

so the required area integral is

$$\begin{aligned}\int_S f(x,y)\,dA &= \int_{\phi=0}^{\phi=\pi/2}\left(\int_{r=1}^{r=2} r^3\sin^2\phi\cos\phi\times r\,dr\right)d\phi\\ &= \int_{\phi=0}^{\phi=\pi/2}\left(\int_{r=1}^{r=2} r^4\sin^2\phi\cos\phi\,dr\right)d\phi.\end{aligned}$$

Note that the integrand contains a factor r^4 rather than r^3. The extra factor of r comes from the area element in polar coordinates. Carrying out the integral over r gives

$$\begin{aligned}\int_S f(x,y)\,dA &= \int_{\phi=0}^{\phi=\pi/2}\left[\tfrac{1}{5}r^5\right]_{r=1}^{r=2}\sin^2\phi\cos\phi\,d\phi\\ &= \tfrac{31}{5}\int_{\phi=0}^{\phi=\pi/2}\sin^2\phi\cos\phi\,d\phi.\end{aligned}$$

The integral over ϕ can be done by noting that the integrand is a product of a function of $\sin\phi$ times the derivative of $\sin\phi$ (namely, $\cos\phi$). This suggests that we make the substitution $u=\sin\phi$, giving $du/d\phi=\cos\phi$.

The $\phi = 0$ limit corresponds to $u = \sin 0 = 0$, and the $\phi = \pi/2$ limit corresponds to $u = \sin(\pi/2) = 1$. Putting everything together, we get

$$\begin{aligned}\int_S f(x,y)\,dA &= \tfrac{31}{5}\int_{\phi=0}^{\phi=\pi/2} u^2\,\frac{du}{d\phi}\,d\phi \\ &= \tfrac{31}{5}\int_{u=0}^{u=1} u^2\,du \\ &= \tfrac{31}{5}\times\tfrac{1}{3} = \tfrac{31}{15}.\end{aligned}$$

Note: In this example, the integrand is a *product* of a function of r and a function of ϕ, and the limits of integration are all *constants*. In cases like this, it is legitimate to split the integral into the product of two integrals (just as we did in Cartesian coordinates). We can therefore write

$$\int_S f(x,y)\,dA = \int_{r=1}^{r=2} r^4\,dr \times \int_{\phi=0}^{\phi=\pi/2} \sin^2\phi\cos\phi\,d\phi.$$

Evaluation of these integrals gives the same answer as before: $\frac{31}{5}\times\frac{1}{3} = \frac{31}{15}$.

Exercise 15

A circular laboratory dish of radius R is covered with bacteria. Relative to an origin at the centre of the dish, the *surface number density* of bacteria is given by the function

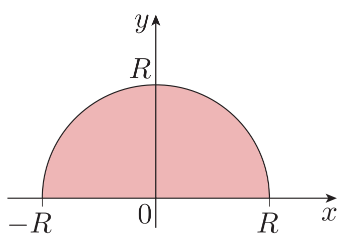
$$f(x,y) = \frac{C}{R^4}\,(2R^2 - x^2 - y^2),$$

where x and y are Cartesian coordinates, and C is a constant. Find the total number of bacteria on the dish.

The surface number density is the number per unit area at a given point. This is modelled as a smoothly-varying function.

Exercise 16

Use polar coordinates to evaluate the area integrals $\int_S x\,dA$ and $\int_S y\,dA$, where S is the semicircular area shown below.



Exercise 17

The function $f(r,\phi) = e^{-r^2}$ is expressed in polar coordinates. Evaluate the area integral of this function over the entire xy-plane.

(*Hint*: An area integral over the entire xy-plane can be carried out in polar coordinates by letting ϕ range from 0 to 2π, and r range from 0 to ∞.)

3.2 Volume integrals in cylindrical coordinates

Cylindrical coordinates are a natural extension of polar coordinates to three dimensions. This subsection introduces cylindrical coordinates and uses them to evaluate volume integrals.

The alternative term **cylindrical polar coordinates** is also used.

Figure 27 shows how the **cylindrical coordinates** (r, ϕ, z) of a point P are defined.

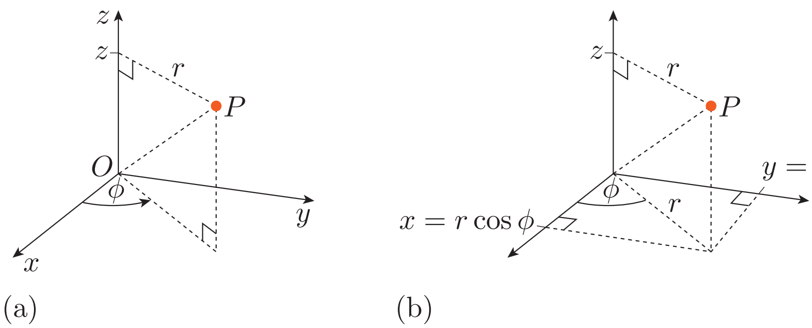


Figure 27 A cylindrical coordinate system: (a) coordinates (r, ϕ, z); (b) relationship to Cartesian coordinates

Note carefully: in cylindrical coordinates, r is *not* defined as the distance of P from the origin O.

- The **radial coordinate** r is the perpendicular distance from the z-axis to P. It ranges from 0 on the z-axis to infinity.
- The **angular coordinate** ϕ is the angle (measured in radians) between the positive x-axis and the projection of OP onto the xy-plane. The allowed range of ϕ corresponds to a complete circuit, and may be taken to be between 0 and 2π. The sense of increasing ϕ is as shown in the diagram (anticlockwise when viewed from a point on the positive z-axis).
- The **axial coordinate** z is identical to the z-coordinate of Cartesian coordinates.

In effect, cylindrical coordinates use the Cartesian z-coordinate parallel to the z-axis and polar coordinates perpendicular to the z-axis. Using Figure 27(b), it is easy to see that a point with cylindrical coordinates (r, ϕ, z) has the following Cartesian coordinates.

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z. \tag{21}$$

To carry out a volume integral in cylindrical coordinates, we must first construct a volume element and find an expression for its volume.
A suitable volume element is shown in Figure 28(a). This is obtained by starting at a point P with cylindrical coordinates (r, ϕ, z), and making small positive increments δr, $\delta\phi$ and δz in each of these coordinates. The edges of the volume element are formed by *coordinate lines* – that is, lines or curves along which *one* cylindrical coordinate increases while the other two coordinates have constant values. In Figure 28, the green, red and blue curves correspond to small increases in r, ϕ and z, respectively.
A wider perspective of these coordinate lines is shown in Figure 28(b).

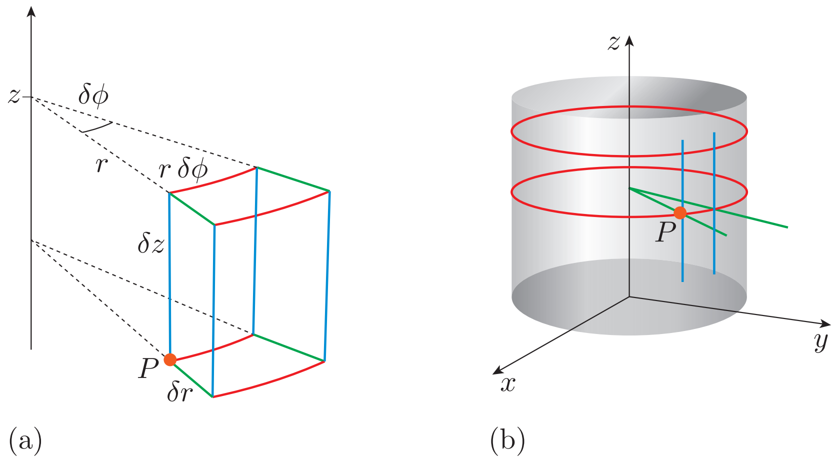


Figure 28 A volume element in cylindrical coordinates: (a) a close-up view; (b) a wider perspective

As its dimensions become very small, the volume element approaches a cuboid in shape. You can see from Figure 28(a) that this cuboid has sides of length δr, $r\,\delta\phi$ and δz. The element therefore has volume

$$\delta V = r\,\delta r\,\delta\phi\,\delta z. \tag{22}$$

Recall that a circular arc of radius r subtending an angle $\delta\phi$ (in radians) has length $r\,\delta\phi$.

Now, suppose that $f(r,\phi,z)$ is a function of cylindrical coordinates. Then the volume integral of f over any given region is approximated by adding contributions of the form

$$f(r,\phi,z)\,\delta V \simeq f(r,\phi,z)\,r\,\delta r\,\delta\phi\,\delta z$$

from all the volume elements that make up the region. We do this in the limit of an infinite number of infinitesimally small volume elements; then our approximations become exact, and at the same time the sum becomes a volume integral. If $f(r,\phi,z)$ is the density (the mass per unit volume) inside a given region, this volume integral gives the total mass contained in the region.

To take a definite case, suppose that the region of integration is a cylinder of radius R and height h, aligned on the z-axis and with its base in the xy-plane (see Figure 29). Then the limits for the r-integral are $r = 0$ and $r = R$, the limits for the ϕ-integral are $\phi = 0$ and $\phi = 2\pi$, and the limits for the z-integral are $z = 0$ and $z = h$. The volume integral of $f(r,\phi,z)$ over this cylinder can therefore be written as follows.

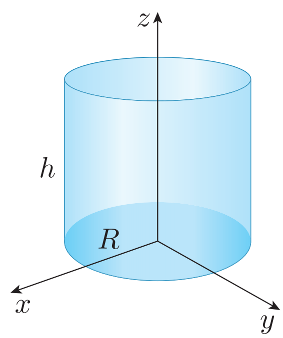


Figure 29 A cylindrical region D

Volume integral over a cylinder in cylindrical coordinates

The volume integral of $f(r,\phi,z)$ over the cylindrical region D in Figure 29 is given by

$$\int_D f(r,\phi,z)\,dV = \int_{z=0}^{z=h}\left(\int_{\phi=0}^{\phi=2\pi}\left(\int_{r=0}^{r=R} f(r,\phi,z)\,r\,dr\right)d\phi\right)dz. \tag{23}$$

The factor r inside the integral must not be forgotten!

The brackets show that we have chosen to integrate first over r, then over ϕ, and finally over z – but any other ordering would be equally valid in this case. For each of the integrations, all variables other than the variable of integration are held constant. For example, ϕ and z are held constant when we integrate over r.

There is nothing unexpected here. The first two integrals, over r and ϕ, correspond to integrating over a planar region in polar coordinates, with z treated as a constant. Only the final integration over z is new.

Example 7

A cylinder of height h and radius R has its central axis along the z-axis, with its base in the xy-plane as in Figure 29. The density of this cylinder is given by

$$f(x, y, z) = \frac{m}{R^5}(x^2 + y^2 + z^2),$$

where x, y and z are Cartesian coordinates, and m is a constant. Find the total mass of the cylinder in terms of m, h and R. Evaluate your answer in the special case where $h = 2R$.

Solution

The total mass of the cylinder is given by the volume integral

$$M = \int_{\text{cylinder}} f(x, y, z)\, dV.$$

Because the region of integration is cylindrical in shape, cylindrical coordinates are the natural choice. We therefore need to express the given density function in cylindrical coordinates. Using equations (21), we get

$$f(r, \phi, z) = \frac{m}{R^5}(r^2 \cos^2\phi + r^2 \sin^2\phi + z^2) = \frac{m}{R^5}(r^2 + z^2),$$

where, as usual, we have used the same symbol f for the density function irrespective of the coordinate system.

The total mass is then given by the integral

Any multiplicative *constants*, such as m/R^5, can be taken outside all the integral signs.

$$\begin{aligned} M &= \int_{z=0}^{z=h} \left(\int_{\phi=0}^{\phi=2\pi} \left(\int_{r=0}^{r=R} \frac{m}{R^5}(r^2 + z^2)\, r\, dr \right) d\phi \right) dz \\ &= \frac{m}{R^5} \int_{z=0}^{z=h} \left(\int_{\phi=0}^{\phi=2\pi} \left(\int_{r=0}^{r=R} (r^3 + z^2 r)\, dr \right) d\phi \right) dz. \end{aligned}$$

Integrating over r and applying the limits of integration gives

$$M = \frac{m}{R^5} \int_{z=0}^{z=h} \left(\int_{\phi=0}^{\phi=2\pi} \left(\tfrac{1}{4}R^4 + \tfrac{1}{2}R^2 z^2 \right) d\phi \right) dz.$$

The integration over ϕ is easy. It gives a factor of 2π, which can be taken outside the integral. So

$$M = \frac{2\pi m}{R^5} \int_{z=0}^{z=h} \left(\tfrac{1}{4}R^4 + \tfrac{1}{2}R^2 z^2 \right) dz.$$

Finally, the integral over z gives

$$M = \frac{2\pi m}{R^5}\left(\tfrac{1}{4}R^4 h + \tfrac{1}{6}R^2 h^3\right).$$

In the special case where $h = 2R$, the mass of the cylinder is

$$M = \frac{2\pi m}{R^5}\left(\tfrac{1}{2}R^5 + \tfrac{4}{3}R^5\right) = \tfrac{11}{3}\pi m.$$

It is sensible to use cylindrical coordinates for all shapes based on cylinders – such as hollow cylinders or segments of a cylinder. Of course, the limits of integration must be adjusted for each particular case.

Exercise 18

A hollow cylinder, with its central axis of symmetry along the z-axis, has inner radius 2 and outer radius 5. The two flat ends of the cylinder are at $z = -1$ and $z = +1$. Find the volume integral of the function $f(r, \phi, z) = rz^2$ over the volume of this hollow cylinder, where r is the distance from the z-axis.

Volumes with axial symmetry

Consider the shape in Figure 30. This shape is unchanged if we rotate it through any angle around the red axis. Such a shape is said to have **axial symmetry**, and the red axis is called the **axis of symmetry**. If a volume integral is over a region with axial symmetry, it is generally advisable to use cylindrical coordinates rather than Cartesian coordinates, with the z-axis coincident with the axis of symmetry.

Figure 30 A shape with axial symmetry relative to the red axis

We now consider calculating the volumes of objects with axial symmetry. In general, the limits of integration for these volumes are not all constants, but axial symmetry is a very useful simplifying feature: it means that the limits of integration of the r- and z-integrals cannot depend on ϕ. So the volume V of any axially-symmetric region can be expressed in cylindrical coordinates as

$$V = \int_{\phi=0}^{\phi=2\pi}\left(\int_{z=z_1}^{z=z_2}\left(\int_{r=r_{\min}(z)}^{r=r_{\max}(z)} 1 \times r\,dr\right)dz\right)d\phi. \qquad (24)$$

Here, the functions $r_{\min}(z)$ and $r_{\max}(z)$ give the minimum and maximum values of the radial coordinate r at a given value of z. If the object is hollow around the z-axis, then $r_{\min}(z)$ is non-zero for at least some values of z, but a solid object has $r_{\min}(z) = 0$ for all z. The values $z = z_1$ and $z = z_2$ are the minimum and maximum values of the z-coordinate in the object.

Recall that in cylindrical coordinates, the radial coordinate is the distance from the z-axis.

We can complete the inner integral over r in equation (24) to obtain

$$V = \int_{\phi=0}^{\phi=2\pi}\left(\int_{z=z_1}^{z=z_2} \tfrac{1}{2}\left(r_{\max}^2(z) - r_{\min}^2(z)\right)dz\right)d\phi. \qquad (25)$$

The remaining limits of integration are all constants, so we can reverse the order of the integrals to get

$$V = \int_{z=z_1}^{z=z_2} \left(\int_{\phi=0}^{\phi=2\pi} \tfrac{1}{2} \left(r_{\text{max}}^2(z) - r_{\text{min}}^2(z) \right) d\phi \right) dz.$$

Integrating over ϕ and taking constants outside the remaining integral, we get the following result.

Volume of an axially symmetric object

$$V = \pi \int_{z=z_1}^{z=z_2} \left(r_{\text{max}}^2(z) - r_{\text{min}}^2(z) \right) dz. \tag{26}$$

In the special case of a solid object, $r_{\text{min}}(z) = 0$ for all z, so

$$V = \pi \int_{z=z_1}^{z=z_2} r_{\text{max}}^2(z)\, dz. \tag{27}$$

This formula can be interpreted by imagining that the object is made up of many thin discs, each of thickness δz, stacked one on top of the other. For a given value of z, the appropriate disc has radius $r_{\text{max}}(z)$, area $\pi r_{\text{max}}^2(z)$ and volume $\pi r_{\text{max}}^2(z)\,\delta z$. Adding together the volumes of all the discs and taking the limit as δz tends to zero, we recover equation (27).

The following example shows how this result is used.

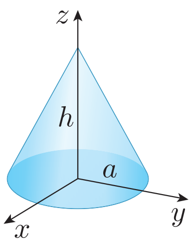


Figure 31 A cone of height h and base radius a

Example 8

Use cylindrical coordinates to find the volume of the cone in Figure 31. This cone has height h and base radius a; its axis of symmetry is the z-axis, and its base lies in the xy-plane.

Solution

Figure 32 shows a cross-section through the central axis of the cone. At a given value of z, the surface of the cone has radial coordinate $r = r_{\text{max}}(z)$, as shown in the figure. From similar triangles we see that

$$\frac{h-z}{r_{\text{max}}(z)} = \frac{h}{a}, \quad \text{so} \quad r_{\text{max}}(z) = a\left(1 - \frac{z}{h}\right).$$

Using equation (27), the volume of the cone is

$$V = \pi a^2 \int_{z=0}^{z=h} \left(1 - \frac{z}{h}\right)^2 dz.$$

This integral can be done in a variety of ways. We choose to make the substitution $u = 1 - z/h$, so that $z = h - hu$. Then $dz/du = -h$, and we can make the replacement $dz = -h\,du$. The limits $z = 0$ and $z = h$ correspond to $u = 1$ and $u = 0$, respectively. So the integral becomes

$$V = \pi a^2 \int_{u=1}^{u=0} u^2(-h\,du) = -\pi a^2 h \left[\tfrac{1}{3}u^3\right]_{u=1}^{u=0} = \tfrac{1}{3}\pi a^2 h.$$

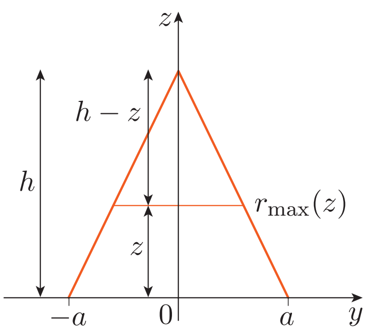


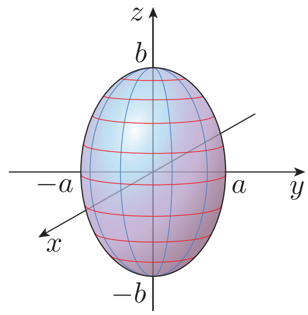
Figure 32 A cross-section through the cone

Exercise 19

A rugby ball has an axis of symmetry along the z-axis, as shown in the margin. In cylindrical coordinates, its surface can be modelled by the equation

$$r_{\text{max}} = a\sqrt{1 - z^2/b^2},$$

where a and b are positive constants. The smallest and largest values of z on the surface of the ball are $z = -b$ and $z = b$. Use equation (27) to find the volume of the ball.



Exercise 20

A sphere of radius R, centred on the origin, is sliced across by a horizontal plane $z = R/2$. The part of the sphere above the plane is a spherical cap. What is the volume of this spherical cap?

3.3 Volume integrals in spherical coordinates

Finally, we discuss **spherical coordinates**, which are used extensively throughout the physical sciences. Figure 33 shows how the spherical coordinates (r, θ, ϕ) of a point P are defined.

The alternative term **spherical polar coordinates** is also used.

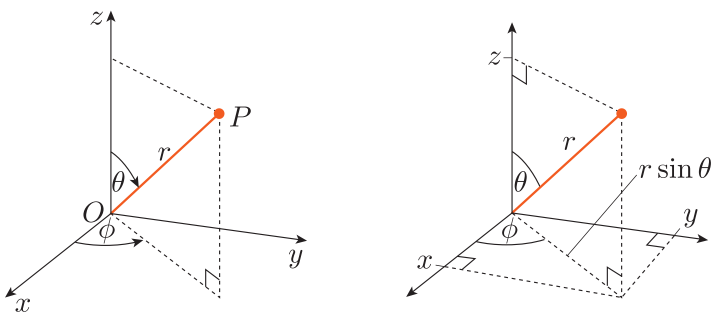


Figure 33 A spherical coordinate system: (a) coordinates (r, θ, ϕ); (b) relationship to Cartesian coordinates

- The **radial coordinate** r is the distance of the point from the origin O. It ranges from 0 at the origin to infinity, and is never negative. Note that this radial coordinate is *not* the same as the radial coordinate in cylindrical coordinates. It is therefore always important to state which coordinate system is being used.
- The **polar angle** θ is the smaller of the angles (in radians) between the positive z-axis and the line OP. It ranges from 0 along the positive z-axis to π along the negative z-axis.
- The **azimuthal angle** ϕ is the angle (in radians) between the positive x-axis and the projection of OP in the xy-plane. It increases in the sense shown, and is the same as the angular coordinate ϕ in cylindrical coordinates. The allowed range of ϕ corresponds to a complete circuit, and may be taken to lie between 0 and 2π.

At first sight, it may seem surprising that θ does not range from 0 to 2π. To see why this is so, consider the Earth with $\theta = 0$ at the North pole and $\theta = \pi$ at the South pole. Then it is clear that all latitudes are covered by letting θ range from 0 to π, and all longitudes are covered by letting ϕ range from 0 to 2π. If we gave θ a range larger than $0 \le \theta \le \pi$, we would be in danger of 'double-counting' in volume or surface integrals.

Spherical coordinates (r, θ, ϕ) can be related to Cartesian coordinates (x, y, z) using the trigonometry of right-angled triangles in Figure 33(b).

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta. \tag{28}$$

These equations can be used to express any function of x, y and z in terms of r, θ and ϕ. For example,

$$\begin{aligned} z^2 - (x^2 + y^2) &= r^2\cos^2\theta - (r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi) \\ &= r^2\cos^2\theta - r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) \\ &= r^2(\cos^2\theta - \sin^2\theta) \\ &= r^2\cos(2\theta). \end{aligned}$$

Recall that $\sin^2\phi + \cos^2\phi = 1$ and $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

To carry out a volume integral in spherical coordinates, we need to identify an appropriate volume element and find an expression for its volume. The required volume element is shown in Figure 34(a). This is obtained by starting at a point P with spherical coordinates (r, θ, ϕ), and making small positive increments δr, $\delta\theta$ and $\delta\phi$ in each of these coordinates.

- The edge PQ corresponds to an increase in r (with θ and ϕ constant).
- The edge PR corresponds to an increase in θ (with r and ϕ constant).
- The edge PS corresponds to an increase in ϕ (with r and θ constant).

A wider perspective is shown in Figure 34(b), where the curves along which a single spherical coordinate varies are shown in green, blue and red for r, θ and ϕ, respectively.

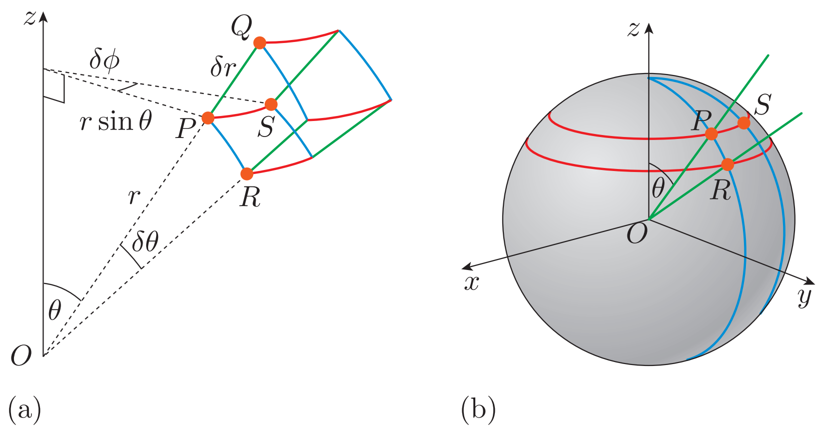


Figure 34 A volume element in spherical coordinates: (a) a close-up view; (b) a wider perspective

As its dimensions become very small, the volume element approaches a cuboid in shape, so its volume δV is given by

$$\delta V \simeq PQ \times PR \times PS.$$

The length of PQ is simply δr. In Figure 34(a), PR is an arc of a blue circle of radius r. This arc is generated by an angular change $\delta\theta$ in θ, so its length is $r\,\delta\theta$. Finally, PS is an arc of a red circle. Using the trigonometry in Figure 34(a), you can see that the radius of this circle is $r\sin\theta$. The arc PS is generated by an angular change $\delta\phi$ in ϕ, so its length is $r\sin\theta\,\delta\phi$. We therefore have

$$\delta V \simeq \delta r \times r\,\delta\theta \times r\sin\theta\,\delta\phi = r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi. \tag{29}$$

The volume integral over a given region is then found in the usual way: we cover the region with tiny volume elements, then take the limit as the number of elements increases and the volume of each element tends to zero. A sum over volume elements then becomes a volume integral, with limits of integration appropriate for the given region. Here is the result for a sphere.

Volume integral over a sphere in spherical coordinates

The volume integral of $f(r,\theta,\phi)$ over a spherical region of radius R, centred on the origin, is given by

$$I = \int_{\phi=0}^{\phi=2\pi} \left(\int_{\theta=0}^{\theta=\pi} \left(\int_{r=0}^{r=R} f(r,\theta,\phi)\, r^2 \sin\theta\, dr \right) d\theta \right) d\phi. \tag{30}$$

Do not forget the factor $r^2\sin\theta$ inside the integral. It appears in all volume integrals based on spherical coordinates.

This result is easily adapted to other regions by changing the limits of integration. For example, a hollow spherical shell corresponds to taking $R_1 \le r \le R_2$, and a hemisphere with $z \ge 0$ is obtained by taking $0 \le \theta \le \pi/2$. Spherical coordinates are particularly useful when the limits in all three integrals are constants.

If the function $f(r,\theta,\phi)$ represents density (the mass per unit volume), then the volume integral is the total mass in the given region. If $f(r,\theta,\phi) = 1$, then the volume integral is the volume of the region. For example, the volume of a hollow spherical shell with inner radius R_1 and outer radius R_2 is given by

$$V = \int_{\phi=0}^{\phi=2\pi} \left(\int_{\theta=0}^{\theta=\pi} \left(\int_{r=R_1}^{r=R_2} r^2 \sin\theta\, dr \right) d\theta \right) d\phi.$$

As always, we work from the inside outwards. The first integration is over r, with θ and ϕ held constant. This gives

$$\begin{aligned} V &= \int_{\phi=0}^{\phi=2\pi} \left(\int_{\theta=0}^{\theta=\pi} \left[\tfrac{1}{3} r^3 \sin\theta\right]_{r=R_1}^{r=R_2} d\theta \right) d\phi \\ &= \frac{R_2^3 - R_1^3}{3} \int_{\phi=0}^{\phi=2\pi} \left(\int_{\theta=0}^{\theta=\pi} \sin\theta\, d\theta \right) d\phi. \end{aligned}$$

The remaining angular integrals are easily done. We get

$$V = \frac{R_2^3 - R_1^3}{3} \int_{\phi=0}^{\phi=2\pi} \left[-\cos\theta\right]_{\theta=0}^{\theta=\pi} d\phi$$
$$= \frac{R_2^3 - R_1^3}{3} \int_{\phi=0}^{\phi=2\pi} 2\, d\phi = \tfrac{4}{3}\pi(R_2^3 - R_1^3).$$

This is the difference between the volume of a solid sphere of radius R_2 and the volume of a solid sphere of radius R_1, just as you would expect. The calculation is a good advertisement for using spherical coordinates in spherically symmetric situations; going through the lengthy chore of calculating the volume of a sphere in Cartesian coordinates is not the smart option!

The ordering of the integrals in equation (30) implies that we integrate first over r, then over θ, and finally over ϕ. However, if the limits of integration are all constants – as they are in equation (30) – we can write the integrals in a different order and still get the same answer. The choice of order can sometimes affect the ease of the integrations.

Example 9

In spherical coordinates, a function of position takes the form

$$f(r,\theta,\phi) = \frac{1}{(a^2 + r^2 - 2ar\cos\theta)^{1/2}},$$

where a is a positive constant. Integrate this function over the volume of a sphere of radius $R < a$, centred on the origin.

(*Hint*: You may find the following integral useful:

This integral can be confirmed by differentiating the result.

$$\int \frac{\sin\theta}{(a^2 + b^2 - 2ab\cos\theta)^{1/2}}\, d\theta = \frac{1}{ab}(a^2 + b^2 - 2ab\cos\theta)^{1/2} + C,$$

where a and b are constants, and C is an arbitrary constant of integration.)

Solution

In spherical coordinates we write the volume integral in the form

$$I = \int_{r=0}^{r=R} \left(\int_{\phi=0}^{\phi=2\pi} \left(\int_{\theta=0}^{\theta=\pi} \frac{r^2 \sin\theta}{(a^2 + r^2 - 2ar\cos\theta)^{1/2}}\, d\theta \right) d\phi \right) dr,$$

where the factor $r^2 \sin\theta$ comes from the expression for the volume element.

Here, we have chosen to integrate over θ first, then over ϕ, and finally over r. Our motivation for tackling the integrals in this order is that the task of integrating $f(r,\theta,\phi)$ over r looks tough. We integrate over the angles first, in the hope that things will get easier!

Using the symbol J to denote the integral over θ, we have

$$J = \int_{\theta=0}^{\theta=\pi} \frac{r^2 \sin\theta}{(a^2 + r^2 - 2ar\cos\theta)^{1/2}}\, d\theta.$$

Holding r constant and using the integral given in the question with $b = r$, we get

$$\begin{aligned}
J &= \left[\frac{r^2}{ar}(a^2 + r^2 - 2ar\cos\theta)^{1/2}\right]_{\theta=0}^{\theta=\pi} \\
&= \frac{r}{a}\left((a^2 + r^2 + 2ar)^{1/2} - (a^2 + r^2 - 2ar)^{1/2}\right) \\
&= \frac{r}{a}\left(((a+r)^2)^{1/2} - ((a-r)^2)^{1/2}\right).
\end{aligned}$$

Within the sphere, we know that $0 \leq r \leq R$, and the question tells us that $R < a$, so we have $0 \leq r < a$. Hence the appropriate square roots are

$$((a+r)^2)^{1/2} = a + r \quad \text{and} \quad ((a-r)^2)^{1/2} = a - r,$$

giving

$$J = \frac{r}{a}\big((a+r) - (a-r)\big) = \frac{2r^2}{a}.$$

Our gamble has paid off: the integration over θ has simplified things considerably. The volume integral then becomes

$$I = \int_{r=0}^{r=R}\left(\int_{\phi=0}^{\phi=2\pi}\frac{2r^2}{a}\,d\phi\right)dr = \int_{r=0}^{r=R}\frac{4\pi r^2}{a}\,dr = \frac{1}{a}\,\frac{4\pi R^3}{3}.$$

A groundbreaking discovery

Using some additional arguments, physicists can use the result of Example 9 to show that the gravitational effect of a uniform sphere of mass M, measured at any point outside the sphere, is the same as that of a particle of mass M placed at the centre of the sphere.

This discovery was of great historic importance. As early as 1666, Isaac Newton took the gravitational effect of a sphere as being equivalent to a particle placed at its centre, but initially assumed this to be a crude working approximation. Nearly twenty years later, in 1685, he finally succeeded in proving the fact. (Newton's calculation followed a slightly different route to that given here, but the physical conclusions are the same.) We know from his own words that he had no expectation of so beautiful a result until it emerged from his mathematical investigation. The discovery was groundbreaking; it removed the barrier to precise astronomical calculations, and the next year Newton felt able to publish his masterpiece, *Philosophiae Naturalis Principia Mathematica* (Figure 35).

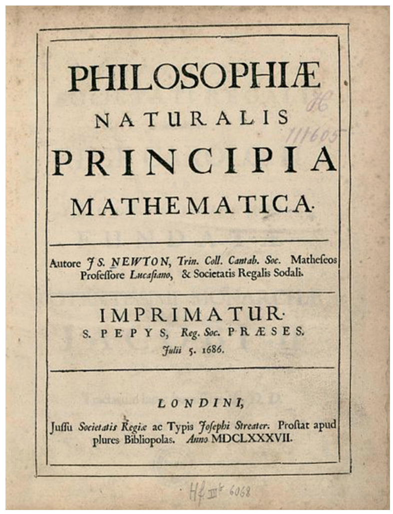


Figure 35 The title page of Newton's masterpiece, usually known as the *Principia*

Frequently, the limits of integration are constants *and* the function to be integrated is a product of single-variable functions of r, θ and ϕ:

$$f(r,\theta,\phi) = u(r)\,v(\theta)\,w(\phi).$$

Under these circumstances, we can write the volume integral in equation (30) as the product of three ordinary integrals:

$$I = \int_{r=0}^{r=R} u(r)\,r^2\,dr \times \int_{\theta=0}^{\theta=\pi} v(\theta)\sin\theta\,d\theta \times \int_{\phi=0}^{\phi=2\pi} w(\phi)\,d\phi.$$

Example 10

Find the mass of a sphere of radius R, centred on the origin, with a density function given by $f(x,y,z) = mz^2/R^5$, where m is a constant. Convert from Cartesian coordinates to spherical coordinates before carrying out an appropriate volume integral.

Solution

Using equations (28), the density function in spherical coordinates is

$$f(r,\theta,\phi) = \frac{m}{R^5}\,r^2\cos^2\theta,$$

so the mass of the sphere is

$$M = \int_{\phi=0}^{\phi=2\pi}\left(\int_{\theta=0}^{\theta=\pi}\left(\int_{r=0}^{r=R}\frac{mr^2}{R^5}\cos^2\theta\times r^2\sin\theta\,dr\right)d\theta\right)d\phi.$$

The integrand is a product function, and the limits of integration are all constants, so the volume integral can be split into a product of three ordinary integrals:

$$M = \frac{m}{R^5}\int_{\phi=0}^{\phi=2\pi}1\,d\phi\times\int_{\theta=0}^{\theta=\pi}\cos^2\theta\sin\theta\,d\theta\times\int_{r=0}^{r=R}r^4\,dr.$$

The integral over θ can be done by making the substitution $u=\cos\theta$. Then $du/d\theta = -\sin\theta$, and the new lower and upper limits are $u=1$ and $u=-1$, respectively. Hence

$$\int_{\theta=0}^{\theta=\pi}\cos^2\theta\sin\theta\,d\theta = \int_{\theta=0}^{\theta=\pi}u^2\left(-\frac{du}{d\theta}\right)d\theta = -\int_{u=1}^{u=-1}u^2\,du = \tfrac{2}{3}.$$

The remaining integrals over ϕ and r are easily done. Collecting everything together, we get

$$M = \frac{m}{R^5}\times 2\pi\times\tfrac{2}{3}\times\tfrac{1}{5}R^5 = \tfrac{4}{15}\pi m.$$

Exercise 21

The function

$$f(r,\theta,\phi) = \frac{\sin\theta}{r}\quad (r\neq 0)$$

is expressed in spherical coordinates. Find the volume integral of this function over the region between two concentric spheres, centred on the origin, and of radii $r=1$ and $r=2$.

Exercise 22

(a) Given a function $f(r)$, where r is the distance from the origin, show that the volume integral of $f(r)$ over a sphere of radius R, centred on the origin, can be expressed as

$$\int_{\text{sphere}} f\,dV = 4\pi\int_0^R f(r)\,r^2\,dr.$$

(b) Use your answer to part (a) to find the volume integral of

$$g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

over a sphere of radius R, centred on the origin.

4 A review of coordinate systems

You have now met several coordinate systems:

- Cartesian coordinates in two and three dimensions
- polar coordinates in two dimensions
- cylindrical coordinates in three dimensions
- spherical coordinates in three dimensions.

In each case, we defined area or volume elements, and used these elements to calculate area and volume integrals. All of these area and volume elements can be treated from a unified point of view, using the concept of *scale factors*. The basic concept of a scale factor is explained in Subsection 4.1, while Subsection 4.2 gives further details.

4.1 Orthogonal coordinate systems and scale factors

In any given coordinate system, we can define lines or curves along which just one coordinate varies, while the other coordinates remain fixed. Such lines or curves are called **coordinate lines**.

For example, in polar coordinates (r, ϕ), there are r-coordinate lines and ϕ-coordinate lines (Figure 36). The r-coordinate lines are shown in blue: these are radial lines along which r varies and ϕ has a constant value. The ϕ-coordinate lines are shown in orange: these are circles around which ϕ varies and r has a constant value.

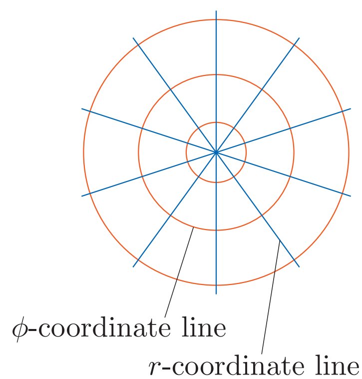


Figure 36 Coordinate lines in polar coordinates

Each coordinate has a corresponding coordinate line. In Cartesian coordinates, the x-, y- and z-coordinate lines are all straight lines, parallel to the axes. In polar, cylindrical and spherical coordinate systems, at least one type of coordinate line is not straight; for this reason they are often described as being **curvilinear coordinate systems**.

All the coordinate systems discussed so far share an important property. In each system, *the coordinate lines corresponding to different coordinates meet at right angles*. For example, in Cartesian coordinates, the x-, y- and z-coordinate lines are perpendicular to one another. In polar coordinates, the radial r-coordinate lines are perpendicular to the circular ϕ-coordinate lines, and so on. Coordinate systems with this property are said to be *orthogonal*.

> **Orthogonal coordinate systems**
>
> A coordinate system is said to be **orthogonal** if its coordinate lines, corresponding to different coordinates, meet at right angles.
>
> Cartesian, polar, cylindrical and spherical coordinate systems are all orthogonal.

Now consider the process of forming an area or volume element in an orthogonal coordinate system. First, we review the argument for polar coordinates.

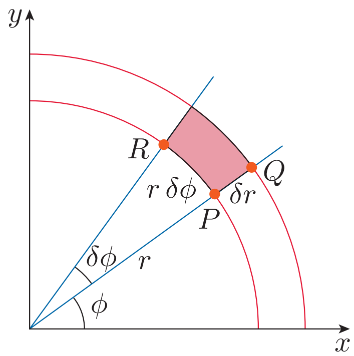


Figure 37 An area element in polar coordinates (r, ϕ)

Figure 37 shows a tiny area element in this coordinate system at a point P with polar coordinates (r, ϕ). This element has adjacent sides PQ and PR, and its size has been exaggerated for clarity. Note that PQ is part of an r-coordinate line, and PR is part of a ϕ-coordinate line. Because the polar coordinate system is orthogonal, PQ and PR meet at right angles. This is a great simplifying feature because it means that the tiny area element can be approximated by a rectangle. Such an element has area

$$\delta A = PQ \times PR.$$

The lengths PQ and PR are easily found from Figure 37. We have $PQ = \delta r$ and $PR = r\,\delta\phi$, so we conclude that

$$\delta A = r\,\delta r\,\delta\phi.$$

Note that this area element is not just the product of the coordinate increments δr and $\delta\phi$. The angular increment $\delta\phi$ is dimensionless, and it must be multiplied by the factor r to produce the length $PR = r\,\delta\phi$. That is why the formula for δA contains a factor of r.

Now consider a more general case. Suppose that we have a two-dimensional coordinate system with coordinates (u, v). To keep the argument general, we do not specify the nature of these coordinates – they could be polar coordinates (r, ϕ), or some other choice, but we do insist that the coordinate system is *orthogonal*. This means that the u- and v-coordinate lines (the curves along which just one coordinate varies) meet at right angles.

Starting from a given point with coordinates (u, v), we can make a small increment in u, with v held constant. This small increment generates a small step along the u-coordinate line. However, the length of this step need not be equal to δu. You saw this in the case of polar coordinates, where an increment $\delta\phi$ generates a step of length $r\,\delta\phi$. To deal with this point in a general way, we introduce the concept of a scale factor.

Scale factors

For any coordinate u, the length of the segment of the u-coordinate line between u and $u + \delta u$, where $\delta u > 0$, is expressed as

$$\text{length of segment} = h_u \, \delta u, \tag{31}$$

where h_u is called the **scale factor** for the u-coordinate; this may be a function of the coordinates.

For example, the scale factors for polar coordinates (r, ϕ) are $h_r = 1$ and $h_\phi = r$, corresponding to the segment lengths δr and $r \, \delta\phi$.

Using scale factors, we can write down a general expression for the area of an area element in any orthogonal coordinate system. An area element at a point (u, v) is produced as follows. Starting from the point (u, v) in Figure 38, we step out along the u-coordinate line until u has increased to $u + \delta u$. We also step out along the v-coordinate line until v has increased to $v + \delta v$. This gives two adjacent sides of the area element. From the definition of scale factors, we know that these sides have lengths $h_u \, \delta u$ and $h_v \, \delta v$, respectively. However, the coordinate system is assumed to be *orthogonal*. This means that the coordinate lines meet at right angles, and the area element can be approximated by a rectangle whose area is given by multiplying the lengths of two adjacent sides. We therefore reach the following conclusion.

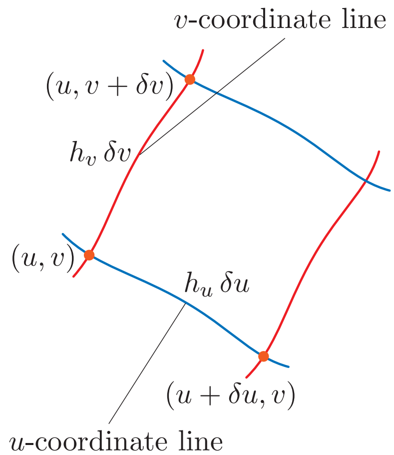


Figure 38 An area element in a general orthogonal coordinate system (u, v)

Area element in orthogonal coordinates

In any orthogonal coordinate system (u, v), an area element has area

$$\delta A = h_u h_v \, \delta u \, \delta v, \tag{32}$$

where h_u and h_v are appropriate scale factors.

Similar ideas apply to orthogonal coordinate systems in three dimensions. There are now three coordinates, (u, v, w). Since the coordinate system is assumed to be orthogonal, the u-, v- and w-coordinate lines meet at right angles, and a tiny volume element can be approximated by a cuboid. The volume of this element is given by multiplying the lengths of three adjacent sides. By the definition of scale factors, these lengths are $h_u \, \delta u$, $h_v \, \delta v$ and $h_w \, \delta w$. So we have the following result.

Volume element in orthogonal coordinates

In any orthogonal coordinate system (u, v, w), a volume element has volume

$$\delta V = h_u h_v h_w \, \delta u \, \delta v \, \delta w, \tag{33}$$

where h_u, h_v and h_v are appropriate scale factors.

The products $h_u h_v$ in two dimensions, and $h_u h_v h_w$ in three dimensions, are called **Jacobian factors**. They occur wherever an area or volume integral uses orthogonal coordinates. Do not make the mistake of leaving them out!

Equations (32) and (33) apply in all orthogonal coordinate systems, but to use these equations in a given coordinate system, you need to know the scale factors. There is a trivial case: in Cartesian coordinates, all the scale factors are equal to 1, and the corresponding area and volume elements are $\delta A = \delta x\, \delta y$ and $\delta V = \delta x\, \delta y\, \delta z$. For other coordinate systems, we can use the results for area and volume elements derived in previous sections to compile a list of all the scale factors that we need.

Scale factors in some orthogonal coordinate systems

Polar coordinates (r, ϕ) $\quad h_r = 1,\ h_\phi = r \quad$ (34)

Cylindrical coordinates (r, ϕ, z) $\quad h_r = 1,\ h_\phi = r,\ h_z = 1 \quad$ (35)

Spherical coordinates (r, θ, ϕ) $\quad h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta \quad$ (36)

The scale factors for polar coordinates correspond to the area element in Figure 37. For reference purposes, the volume elements in cylindrical and spherical coordinates are reproduced in Figure 39, and you can see that these correspond to the scale factors in the list above.

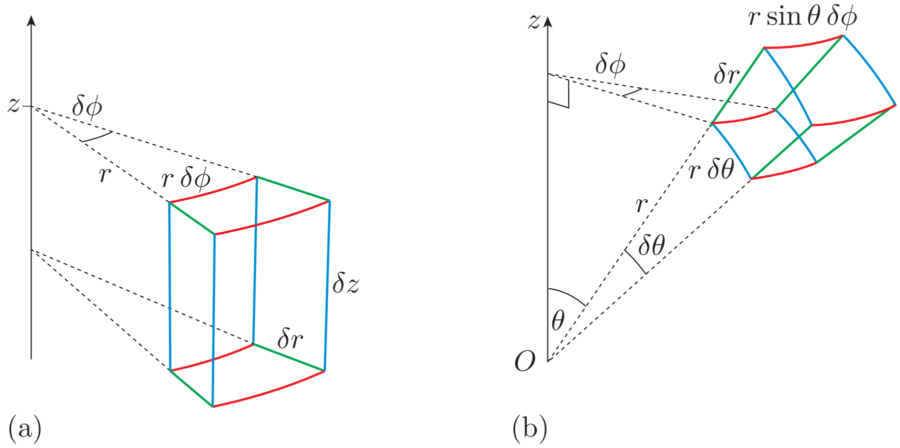


Figure 39 Volume elements in (a) cylindrical coordinates and (b) spherical coordinates

Exercise 23

Use the scale factors in equations (35) and (36) to write down formulas for volume elements in cylindrical and spherical coordinates.

Why should we bother with scale factors? The main reason is conceptual – they provide a unified language for discussing area and volume integrals.

Apart from this, the concepts of *orthogonal coordinate systems*, *coordinate lines* and *scale factors* are all used later in this book. They reappear in contexts other than integration, but the introduction given here provides a good foundation.

4.2 Another way of calculating scale factors

This subsection derives a neat formula (equation (39)) that can be used to find scale factors without drawing diagrams or using trigonometry. The formula is used later, but its derivation will not be assessed.

Suppose that we have a coordinate system (u, v, w), and we know the relationship between Cartesian coordinates (x, y, z) and (u, v, w). For example, in spherical coordinates (r, θ, ϕ), we know that

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi \quad \text{and} \quad z = r\cos\theta.$$

Then we can use this information directly to find the scale factors.

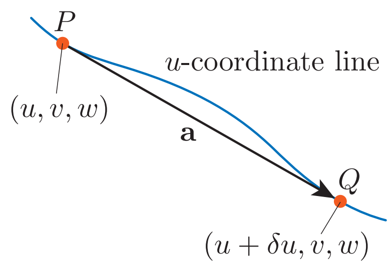


Figure 40 A small displacement along the u-coordinate line

Figure 40 shows the effect of stepping out along the u-coordinate line by making a small increment $\delta u > 0$ in the coordinate u. We move from a point P with coordinates (u, v, w), to a point Q with coordinates $(u + \delta u, v, w)$. The displacement vector between P and Q is denoted by $\mathbf{a}$. The magnitude of this vector is the distance between P and Q, which, by definition, is equal to $h_u\,\delta u$. So we have

$$|\mathbf{a}| = h_u\,\delta u. \tag{37}$$

The displacement vector $\mathbf{a}$ can be written in Cartesian coordinates as

$$\mathbf{a} = \delta x\,\mathbf{i} + \delta y\,\mathbf{j} + \delta z\,\mathbf{k},$$

where δx, δy and δz are the changes in Cartesian coordinates between P and Q. However, the chain rule tells us how a small change in x is related to small changes in u, v and w:

This chain rule was introduced in equation (19) of Unit 7.

$$\delta x = \frac{\partial x}{\partial u}\,\delta u + \frac{\partial x}{\partial v}\,\delta v + \frac{\partial x}{\partial w}\,\delta w = \frac{\partial x}{\partial u}\,\delta u,$$

where the last step follows because $\delta v = 0$ and $\delta w = 0$ along the u-coordinate line. Of course, there are similar expressions for δy and δz, so we conclude that

$$\mathbf{a} = \left(\frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k}\right)\delta u. \tag{38}$$

Since $\delta u > 0$, the magnitude of $\mathbf{a}$ is given by

$$|\mathbf{a}| = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}\,\delta u.$$

Comparing this with equation (37), we get the following general formula for the scale factor h_u.

Formula for a scale factor

$$h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2}. \qquad (39)$$

Similar formulas apply for h_v and h_w, but with partial derivatives taken with respect to v and w, respectively. For a two-dimensional coordinate system (u, v) in the xy-plane, we use essentially the same formula, but with the final term left out.

Example 11

Use equation (39) to find the scale factors for polar coordinates (r, ϕ), which are related to Cartesian coordinates by

$$x = r\cos\phi, \quad y = r\sin\phi.$$

Solution

Taking partial derivatives of x and y with respect to r and ϕ, we have

$$\frac{\partial x}{\partial r} = \cos\phi, \quad \frac{\partial y}{\partial r} = \sin\phi, \quad \frac{\partial x}{\partial \phi} = -r\sin\phi, \quad \frac{\partial y}{\partial \phi} = r\cos\phi.$$

Hence the scale factors are

$$h_r = \sqrt{\cos^2\phi + \sin^2\phi} = 1,$$
$$h_\phi = \sqrt{(-r\sin\phi)^2 + (r\cos\phi)^2} = r.$$

These values agree with those in equation (34).

Exercise 24

Use equation (39) to find the scale factors for spherical coordinates (r, θ, ϕ), which are related to Cartesian coordinates by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

5 Surface integrals

This section considers integrals over surfaces that are not flat. For example, we might know the *surface density* of paint at each point on the surface of a car. This surface density tells us the mass of paint per unit area in the vicinity of any given point on the surface of the car. If we integrate the surface density over the curved surface of the car, we will get the total mass of paint on the car. But how do we integrate over a *curved* surface?

5.1 The surface area of a sphere

As a simple example, consider the surface area of a sphere of radius R. You may know that this surface area is $4\pi R^2$, but it is worth seeing where this comes from.

We choose our origin to be at the centre of the sphere, and set up spherical coordinates (r, θ, ϕ). We need only two coordinates (θ, ϕ) to specify any point on the surface of the sphere (because all such points have $r = R$). We sometimes say that the surface of the sphere is *parametrised* by the coordinates θ and ϕ.

We can draw θ- and ϕ-coordinate lines on the surface of the sphere, and these subdivide the surface into a large number of surface elements (Figure 41). With a very fine subdivision, each tiny element can be approximated by a rectangle with sides of length $h_\theta\,\delta\theta$ and $h_\phi\,\delta\phi$, where $h_\theta = R$ and $h_\phi = R\sin\theta$ are the scale factors for spherical coordinates. Hence the area of an element centred on coordinates (θ, ϕ) is

$$\delta A = h_\theta h_\phi\,\delta\theta\,\delta\phi = R^2\sin\theta\,\delta\theta\,\delta\phi. \tag{40}$$

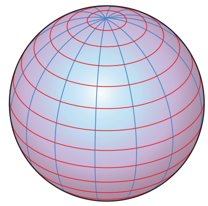


Figure 41 The surface of a sphere is divided into surface elements by a grid formed by θ-coordinate lines (blue) and ϕ-coordinate lines (red)

To find the total surface area of the sphere, we must add the areas of all the surface elements. We do this in the limit of vanishingly small elements, so that the summation is achieved by integration. The values of θ and ϕ cover the ranges $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$, so the total surface area of the sphere is

$$\begin{aligned}\text{surface area} &= \int_{\phi=0}^{\phi=2\pi}\left(\int_{\theta=0}^{\theta=\pi} R^2\sin\theta\,d\theta\right)d\phi\\ &= R^2\int_{\phi=0}^{\phi=2\pi}\Big[-\cos\theta\Big]_{\theta=0}^{\theta=\pi}\,d\phi\\ &= R^2\int_{\phi=0}^{\phi=2\pi} 2\,d\phi = 4\pi R^2,\end{aligned}$$

Remember that R is constant for a given spherical surface.

as expected.

This surface area is the integral over the spherical surface of the function $f = 1$. We can also integrate other functions over this surface. Suppose that the sphere is unevenly coated with a layer whose surface density (i.e. mass per unit area) is

$$f(\theta, \phi) = A\cos^2\theta,$$

where A is a positive constant. Then the total mass of the layer is

$$\begin{aligned}M &= \int_{\phi=0}^{\phi=2\pi}\left(\int_{\theta=0}^{\theta=\pi} f(\theta,\phi)R^2\sin\theta\,d\theta\right)d\phi\\ &= AR^2\int_{\phi=0}^{\phi=2\pi}\left(\int_{\theta=0}^{\theta=\pi}\cos^2\theta\sin\theta\,d\theta\right)d\phi.\end{aligned}$$

The angular integrals over θ and ϕ were calculated in Example 10, and give factors of $\frac{2}{3}$ and 2π, respectively, so we get

$$M = AR^2 \times \tfrac{2}{3} \times 2\pi = \tfrac{4}{3}\pi AR^2.$$

Exercise 25

A sphere of radius R is centred on the origin. Find the surface area of the spherical cap formed by the portion of the sphere that has $z > R/2$.

5.2 A general method for surface integrals

In practice, scientists and engineers do not spend much time evaluating surface integrals. They either consider simple surfaces, such as spheres or cylinders, or use numerical methods.

The calculation of surface integrals on the surface of a sphere works well because points on the surface of a sphere can be labelled by the angular coordinates θ and ϕ of spherical coordinates. This coordinate system is orthogonal, so it generates area elements on the surface of the sphere that can be approximated by tiny rectangles. We know the relevant scale factors h_θ and h_ϕ in this case, so it is fairly easy to obtain equation (40) for an area element.

On a general surface, we have to work harder to get suitable expressions for the area elements. This final subsection develops a general method for doing this, summarised by equations (43) and (44) below. You may be asked to apply these results, but the arguments leading up to them will not be assessed.

Let us assume that the surface under investigation is parametrised by coordinates (u, v). This means that each allowed pair of values (u, v) labels a unique point on the surface. As u and v vary over their allowed ranges, the entire surface is mapped out. You have seen how this works for the surface of a sphere, which is parametrised by the coordinates θ and ϕ, with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Each point on the surface can also be represented by Cartesian coordinates (x, y, z). So there must be some link between Cartesian coordinates and u and v. On the surface, x, y and z will be given by specific functions

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v). \tag{41}$$

For example, in the case of a sphere of radius R, centred on the origin, and parametrised by $\theta = u$ and $\phi = v$, these equations take the form

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta, \tag{42}$$

where R is a constant for a given sphere.

On an arbitrary surface, the u-coordinate lines intermesh with the v-coordinate lines to produce a grid of tiny surface elements (Figure 42).

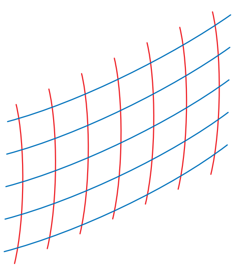


Figure 42 A surface grid formed by intermeshing coordinate lines

In general, the u- and v-coordinate lines do not meet at right angles, and the surface elements are approximated by tiny flat parallelograms, rather than rectangles. We need to find the areas of these parallelograms, one of which is shown greatly enlarged in Figure 43.

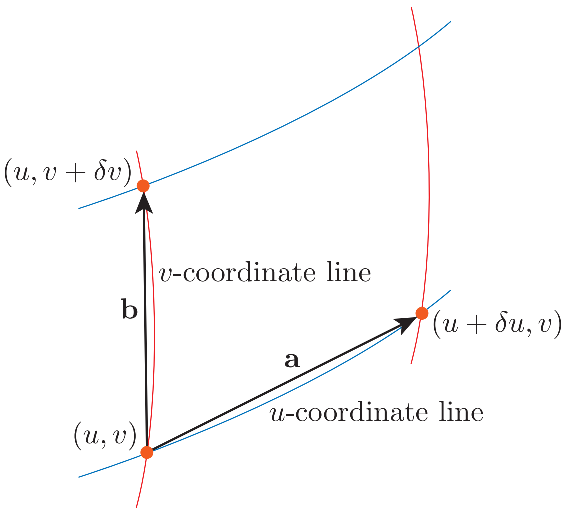


Figure 43 An area element can be approximated by a tiny parallelogram, shown greatly enlarged here

Using equation (34) from Unit 4, we have

$$\text{area of parallelogram} = |\mathbf{a} \times \mathbf{b}|,$$

where $\mathbf{a}$ and $\mathbf{b}$ are the displacement vectors shown in Figure 43. We also know from Unit 4 that the vector product $\mathbf{a} \times \mathbf{b}$ can be expressed as a determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

See Unit 4, equation (61).

However, we obtained an expression for the vector $\mathbf{a}$ in Subsection 4.2. Equation (38) tells us that

$$\mathbf{a} = (a_x, a_y, a_z) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \delta u,$$

and by a similar argument,

$$\mathbf{b} = (b_x, b_y, b_z) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) \delta v.$$

Combining these results with the preceding equations, we obtain the following general result for the area of a surface element on a curved surface.

Area of a surface element on a curved surface

If a surface is parametrised by coordinates (u, v), where $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$, then the area of a surface element is given by

$$\delta A = |\mathbf{J}|\,\delta u\,\delta v, \tag{43}$$

where $|\mathbf{J}|$ is the magnitude of the vector

$$\mathbf{J} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}. \tag{44}$$

The vector $\mathbf{J}$ is sometimes called the **Jacobian vector**.

Note that the vertical lines in equation (43) indicate the magnitude of a vector, while the vertical lines in equation (44) indicate a determinant.

The magnitude of $\mathbf{J}$ determines the area of a surface element. The direction of $\mathbf{J}$ also has a simple interpretation. Because the vector product $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and because $\mathbf{a}$ and $\mathbf{b}$ in Figure 43 lie the plane of the surface, *the vector* $\mathbf{J}$ *is perpendicular to the surface.*

Before using equations (43) and (44) more generally, let us just check that they give the result obtained earlier for the surface element of a sphere, parametrised by θ and ϕ of spherical coordinates. In this case the relevant partial derivatives were calculated in Exercise 24. Setting $r = R$, we get

$$\mathbf{J} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ R\cos\theta\cos\phi & R\cos\theta\sin\phi & -R\sin\theta \\ -R\sin\theta\sin\phi & R\sin\theta\cos\phi & 0 \end{vmatrix}. \tag{45}$$

Exercise 26

Expand the determinant in equation (45), and hence confirm that the area of a surface element on a sphere is given by $\delta A = R^2 \sin\theta\,\delta\theta\,\delta\phi$.

You will need to use the identity $\cos^2 x + \sin^2 x = 1$.

Using the surface element $\delta A = |\mathbf{J}|\,\delta u\,\delta v$, it is easy to write down a general expression for a surface integral.

Surface integral over a curved surface

For a surface S parametrised by coordinates (u, v), the surface integral of a function $f(u, v)$ over S is given by

$$\int_S f\,dA = \int_{v=v_1}^{v=v_2} \left(\int_{u=u_1}^{u=u_2} f(u, v)\,|\mathbf{J}|\,du \right) dv, \tag{46}$$

where the ranges $u_1 \leq u \leq u_2$ and $v_1 \leq v \leq u_2$ are chosen to cover the surface S exactly.

Equation (46) assumes that all the limits of integration are constants. This covers all the cases considered in this module.

To find the area of a surface S, we use equation (46) with $f = 1$.

Example 12

A cone has height h and base radius a (see Figure 44). The sloping surface of this cone can be parametrised by two of the cylindrical coordinates, r and ϕ, with $0 \leq r \leq a$ and $0 \leq \phi \leq 2\pi$. In terms of these parameters, points on the sloping surface of the cone have Cartesian coordinates

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = h\left(1 - \frac{r}{a}\right).$$

Find the area of the sloping surface of the cone (not including its base).

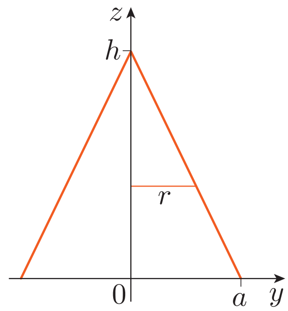


Figure 44 Cross-section through a cone

Solution

Taking partial derivatives of x, y and z with respect to our chosen parameters r and ϕ gives

$$\frac{\partial x}{\partial r} = \cos\phi, \quad \frac{\partial y}{\partial r} = \sin\phi, \quad \frac{\partial z}{\partial r} = -\frac{h}{a},$$
$$\frac{\partial x}{\partial \phi} = -r\sin\phi, \quad \frac{\partial y}{\partial \phi} = r\cos\phi, \quad \frac{\partial z}{\partial \phi} = 0.$$

Using these results in the expression for the Jacobian vector $\mathbf{J}$, we get

$$\begin{aligned}
\mathbf{J} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi & \sin\phi & -h/a \\ -r\sin\phi & r\cos\phi & 0 \end{vmatrix} \\
&= \frac{h}{a} r\cos\phi\,\mathbf{i} + \frac{h}{a} r\sin\phi\,\mathbf{j} + r(\cos^2\phi + \sin^2\phi)\,\mathbf{k} \\
&= \frac{h}{a} r\cos\phi\,\mathbf{i} + \frac{h}{a} r\sin\phi\,\mathbf{j} + r\,\mathbf{k}.
\end{aligned}$$

The square of the magnitude of $\mathbf{J}$ is

$$\begin{aligned}
|\mathbf{J}|^2 &= \frac{h^2}{a^2} r^2(\cos^2\phi + \sin^2\phi) + r^2 \\
&= \left(1 + \frac{h^2}{a^2}\right) r^2,
\end{aligned}$$

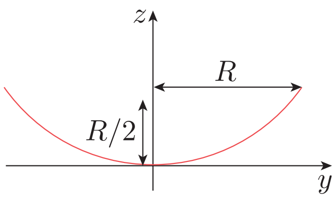
so the area element is

$$\delta A = \left(1 + \frac{h^2}{a^2}\right)^{1/2} r\,\delta r\,\delta\phi.$$

Using this area element and integrating over the ranges of r and ϕ, we get

$$\begin{aligned}
\text{area} &= \left(1 + \frac{h^2}{a^2}\right)^{1/2} \int_{\phi=0}^{\phi=2\pi} \left(\int_{r=0}^{r=a} r\,dr\right) d\phi \\
&= \left(1 + \frac{h^2}{a^2}\right)^{1/2} \times 2\pi \times \frac{a^2}{2} \\
&= \pi a\sqrt{a^2 + h^2}.
\end{aligned}$$

A useful check is provided by letting h tend to zero. The surface area then tends to πa^2, which is the area of a circle, as expected.



Exercise 27

The parabolic reflector dish shown in cross-section in the margin has radius R at its opening and axial depth $R/2$. Its surface can be parametrised by two of the cylindrical coordinates, r and ϕ, with $0 \leq r \leq R$ and $0 \leq \phi \leq 2\pi$. In terms of these parameters, points on the surface of the dish have Cartesian coordinates

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = \frac{r^2}{2R}.$$

Find the surface area of the outside of this dish.

Exercise 28

A certain surface S has parameters u and v, and extends over $0 \leq u \leq 1$ and $0 \leq v \leq 1$. In terms of these parameters, the Cartesian coordinates of points on S are

$$x = \tfrac{1}{2}(u^2 + v^2), \quad y = \tfrac{1}{2}(u^2 - v^2), \quad z = uv.$$

Find the surface area of S.

Learning outcomes

After studying this unit, you should be able to do the following.

- Evaluate area integrals over rectangular and non-rectangular regions of the xy-plane using Cartesian coordinates.
- Evaluate volume integrals over cuboid and non-cuboid regions using Cartesian coordinates.
- Evaluate area and volume integrals using polar, cylindrical and spherical coordinates.
- Define the terms orthogonal coordinate system, coordinate line, scale factor, Jacobian factor and Jacobian vector.
- Evaluate surface integrals over curved surfaces.

Solutions to exercises

Solution to Exercise 1

The integrand is a product of a function of x and a function of y, and the range of integration is a rectangle aligned with the coordinate axes, so we can evaluate the integral as the product of two definite integrals:

$$\begin{aligned}\int_S x^2y^3\,dA &= \left(\int_{x=0}^{x=2} x^2\,dx\right)\times\left(\int_{y=1}^{y=3} y^3\,dy\right)\\ &= \left[\tfrac{1}{3}x^3\right]_{x=0}^{x=2}\times\left[\tfrac{1}{4}y^4\right]_{y=1}^{y=3}\\ &= \frac{8}{3}\times\frac{81-1}{4}=\frac{160}{3}.\end{aligned}$$

Solution to Exercise 2

This integrand does *not* factorise into a function of x times a function of y, so we must evaluate it by two successive integrations. Because the limits for y involve a zero, it is slightly easier to integrate over y first. Remembering to treat x as a constant when we perform the y-integration, we get

$$\begin{aligned}\int_S (1+x+y)\,dA &= \int_{x=1}^{x=4}\left(\int_{y=0}^{y=3}(1+x+y)\,dy\right)dx\\ &= \int_{x=1}^{x=4}\left[y+xy+\tfrac{1}{2}y^2\right]_{y=0}^{y=3}\,dx\\ &= \int_{x=1}^{x=4}\left(\tfrac{15}{2}+3x\right)dx.\end{aligned}$$

Carrying out the remaining integration over x, we conclude that

$$\begin{aligned}\int_S (1+x+y)\,dA &= \left[\tfrac{15}{2}x+\tfrac{3}{2}x^2\right]_{x=1}^{x=4}\\ &= 30+24-\tfrac{15}{2}-\tfrac{3}{2}=45.\end{aligned}$$

Solution to Exercise 3

The required area integral can be written as

$$I=\int_{y=0}^{y=\pi}\left(\int_{x=0}^{x=\pi}\cos(x+y)\,dx\right)dy.$$

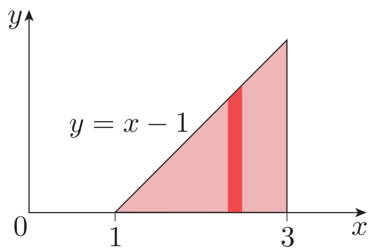
Integrating with respect to x, with y held constant, we obtain

$$\begin{aligned}I &= \int_{y=0}^{y=\pi}\left[\sin(x+y)\right]_{x=0}^{x=\pi}\,dy\\ &= \int_{y=0}^{y=\pi}\big(\sin(\pi+y)-\sin(y)\big)\,dy.\end{aligned}$$

The remaining integral over y gives

$$\begin{aligned}I &= \left[-\cos(\pi+y)+\cos(y)\right]_{y=0}^{y=\pi}\\ &= \big(-\cos(2\pi)+\cos(\pi)\big)-\big(-\cos(\pi)+\cos(0)\big)=-4.\end{aligned}$$

Solution to Exercise 4



The region of integration is shown in the diagram in the margin, with a typical strip parallel to the y-axis marked. It is clear that the lower y-limit is $y = 0$ and the upper y-limit is $y = x - 1$. These limits are the correct way round because $x - 1 > 0$ throughout the region of integration.

We can also immediately read off the lower and upper x-limits as $x = 1$ and $x = 3$, respectively. The required area integral is then given by

$$\int_S (x - y)\, dA = \int_{x=1}^{x=3} \left(\int_{y=0}^{y=x-1} (x - y)\, dy \right) dx.$$

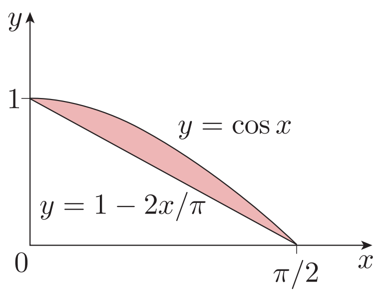
Carrying out the inner integration over y first, we get

$$\begin{aligned}\int_S (x - y)\, dA &= \int_{x=1}^{x=3} \left[xy - \tfrac{1}{2}y^2\right]_{y=0}^{y=x-1} dx \\ &= \int_{x=1}^{x=3} \left(x(x-1) - \tfrac{1}{2}(x-1)^2\right) dx \\ &= \int_{x=1}^{x=3} \left(\tfrac{1}{2}x^2 - \tfrac{1}{2}\right) dx.\end{aligned}$$

Finally, the integral over x gives

$$\begin{aligned}\int_S (x - y)\, dA &= \left[\tfrac{1}{6}x^3 - \tfrac{1}{2}x\right]_{x=1}^{x=3} \\ &= \tfrac{27}{6} - \tfrac{3}{2} - \tfrac{1}{6} + \tfrac{1}{2} \\ &= \tfrac{10}{3}.\end{aligned}$$

Solution to Exercise 5



The figure in the margin shows the required region. We choose to integrate over y first, and so imagine dividing the region into thin vertical strips. The area is then given by

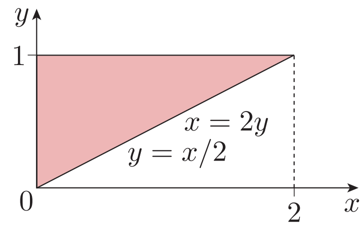
$$I = \int_{x=0}^{x=\pi/2} \left(\int_{y=1-2x/\pi}^{y=\cos x} 1\, dy \right) dx.$$

Carrying out the integrals, we get

$$\begin{aligned}I &= \int_{x=0}^{x=\pi/2} \left[y\right]_{y=1-2x/\pi}^{y=\cos x} dx \\ &= \int_{x=0}^{x=\pi/2} \left(\cos x - 1 + \frac{2x}{\pi}\right) dx \\ &= \left[\sin x - x + \frac{x^2}{\pi}\right]_{x=0}^{x=\pi/2} \\ &= \sin\left(\frac{\pi}{2}\right) - \frac{\pi}{2} + \frac{\pi}{4} \\ &= 1 - \frac{\pi}{4}.\end{aligned}$$

Solution to Exercise 6

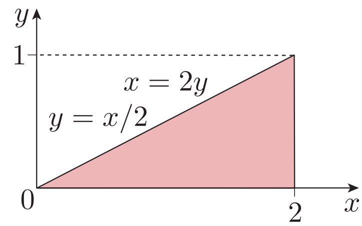
(a) The first diagram in the margin shows the area of integration.



Using this diagram, the given area integral can be written as

$$\int_{y=0}^{y=1}\left(\int_{x=0}^{x=2y} f(x,y)\,dx\right)dy.$$

(b) The second diagram in the margin shows the area of integration.

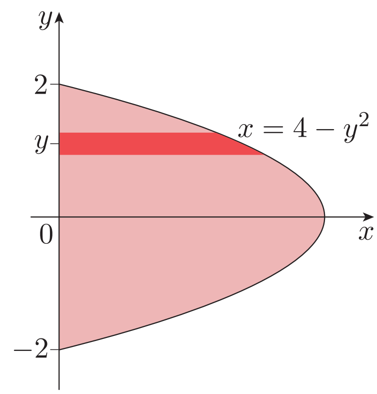


Using this diagram, the given area integral can be written as

$$\int_{y=0}^{y=1}\left(\int_{x=2y}^{x=2} f(x,y)\,dx\right)dy.$$

Solution to Exercise 7

The diagram in the margin shows the given region. A typical narrow horizontal strip extends across this region from $x = 0$ to $x = 4 - y^2$. These are the lower and upper limits of the inner x-integration. The minimum and maximum values of y are $y = -2$ and $y = 2$, and these are the lower and upper limits of the y-integration. So using equations (11) and (12), the area is



$$\begin{aligned}
\text{area} &= \int_{y=-2}^{y=2}\left(\int_{x=0}^{x=4-y^2} 1\,dx\right)dy\\
&= \int_{y=-2}^{y=2}\left[x\right]_{x=0}^{x=4-y^2}\,dy\\
&= \int_{y=-2}^{y=2}(4-y^2)\,dy = \left[4y - \tfrac{1}{3}y^3\right]_{y=-2}^{y=2} = \tfrac{32}{3}.
\end{aligned}$$

Solution to Exercise 8

Following the hint in the question, we integrate first over x, then over y. Considering a horizontal strip, the limits of the x-integration are $0 \le x \le \sqrt{1-y^2}$, so the area integral is

$$\begin{aligned}
\int_S f(x,y)\,dA &= \int_{y=0}^{y=1}\left(\int_{x=0}^{x=\sqrt{1-y^2}} x\,dx\right)dy\\
&= \int_{y=0}^{y=1}\left[\tfrac{1}{2}x^2\right]_{x=0}^{x=\sqrt{1-y^2}}\,dy\\
&= \int_{y=0}^{y=1}\tfrac{1}{2}(1-y^2)\,dy\\
&= \left[\tfrac{1}{2}\left(y - \tfrac{1}{3}y^3\right)\right]_{y=0}^{y=1} = \tfrac{1}{3}.
\end{aligned}$$

You can see *why* it is easier to do the integration in this order: integrating over x first leads to $\frac{1}{2}x^2$, and this allows us to avoid a tricky integral involving square roots. The ability to anticipate such things is a useful skill.

Solution to Exercise 9

Choosing to integrate first over y and then over x, the required area integral is

$$\begin{aligned} I &= \int_{x=0}^{x=1} \left(\int_{y=0}^{y=x} \exp(x^2)\, dy \right) dx \\ &= \int_{x=0}^{x=1} \Big[y\exp(x^2) \Big]_{y=0}^{y=x} dx \\ &= \int_{x=0}^{x=1} x\exp(x^2)\, dx. \end{aligned}$$

This tactic is chosen because the argument x^2 of $\exp(x^2)$ has derivative $2x$, which is proportional to the factor x in the integrand.

We make the substitution $u = x^2$. Then we have $du/dx = 2x$, and the lower and upper limits of integration become $u = 0$ and $u = 1$. Hence

$$\begin{aligned} I &= \int_{x=0}^{x=1} \exp(u)\, \tfrac{1}{2} \frac{du}{dx}\, dx \\ &= \tfrac{1}{2} \int_{u=0}^{u=1} \exp(u)\, du = \tfrac{1}{2} \Big[\exp(u) \Big]_{u=0}^{u=1} = \tfrac{1}{2}(e-1) \simeq 0.859. \end{aligned}$$

If we had tried to integrate first over x and then over y, the area integral would have been written as

$$I = \int_{y=0}^{y=1} \left(\int_{x=y}^{x=1} \exp(x^2)\, dx \right) dy.$$

This is correct, but frustrating, because the integration over x cannot be done using standard mathematical functions.

Solution to Exercise 10

The mass of the block is given by the volume integral

$$\begin{aligned} M &= \int_R (x+y+z)\, dV \\ &= \int_{x=0}^{x=2} \left(\int_{y=1}^{y=2} \left(\int_{z=2}^{z=5} (x+y+z)\, dz \right) dy \right) dx. \end{aligned}$$

Evaluating the inner integral over z gives

$$\begin{aligned} M &= \int_{x=0}^{x=2} \left(\int_{y=1}^{y=2} \Big[xz + yz + \tfrac{1}{2}z^2 \Big]_{z=2}^{z=5} dy \right) dx \\ &= \int_{x=0}^{x=2} \left(\int_{y=1}^{y=2} \left(3x + 3y + \tfrac{21}{2} \right) dy \right) dx. \end{aligned}$$

Evaluating the integral over y gives

$$\begin{aligned} M &= \int_{x=0}^{x=2} \Big[3xy + \tfrac{3}{2}y^2 + \tfrac{21}{2}y \Big]_{y=1}^{y=2} dx \\ &= \int_{x=0}^{x=2} \Big((6x + 6 + 21) - \left(3x + \tfrac{3}{2} + \tfrac{21}{2} \right) \Big) dx \\ &= \int_{x=0}^{x=2} (3x + 15)\, dx. \end{aligned}$$

Finally, the integral over x gives

$$M = \left[\tfrac{3}{2}x^2 + 15x\right]_{x=0}^{x=2} = 36.$$

So the mass of the block is 36 kilograms.

Solution to Exercise 11

We have

$$\begin{aligned} f(x,y,z) &= xyz\, e^{-(x^2+y^2+z^2)} \\ &= xyz\, e^{-x^2} e^{-y^2} e^{-z^2} \\ &= xe^{-x^2} \times ye^{-y^2} \times ze^{-z^2}, \end{aligned}$$

which is a product of the form $u(x)\,v(y)\,w(z)$. The required volume integral over the cube therefore becomes

$$I = \int_{x=0}^{x=1} xe^{-x^2}\,dx \times \int_{y=0}^{y=1} ye^{-y^2}\,dy \times \int_{z=0}^{z=1} ze^{-z^2}\,dz.$$

The individual integrals are evaluated using a method similar to that used in Exercise 9. In the integral over x, for example, we substitute $u = x^2$. Then $du/dx = 2x$, and the limits $x = 0$ and $x = 1$ become $u = 0$ and $u = 1$, so

$$\begin{aligned} \int_{x=0}^{x=1} xe^{-x^2}\,dx &= \int_{x=0}^{x=1} e^{-u}\,\tfrac{1}{2}\,\frac{du}{dx}\,dx \\ &= \int_{u=0}^{u=1} \tfrac{1}{2}e^{-u}\,du \\ &= \left[-\tfrac{1}{2}e^{-u}\right]_{u=0}^{u=1} = \tfrac{1}{2}(1-e^{-1}). \end{aligned}$$

Similar results apply to the y and z integrals, so the volume integral is

$$I = \tfrac{1}{8}(1-e^{-1})^3 \simeq 0.0316.$$

Solution to Exercise 12

The required volume integral is

$$\int_R f(x,y,z)\,dV = \int_{z=0}^{z=1} \left(\int_{y=0}^{y=1-z} \left(\int_{x=0}^{x=1-y-z} z^2\,dx\right) dy\right) dz.$$

The inner integral is with respect to x, and we evaluate this first (holding y and z constant). We get

$$\begin{aligned} \int_{x=0}^{x=1-y-z} z^2\,dx &= \left[z^2 x\right]_{x=0}^{x=1-y-z} \\ &= z^2(1-y-z) = z^2(1-z) - z^2 y. \end{aligned}$$

The middle integral is with respect to y. Evaluating this (with z held constant), we obtain

$$\begin{aligned} \int_{y=0}^{y=1-z} \left(z^2(1-z) - z^2 y\right) dy &= \left[z^2(1-z)y - \tfrac{1}{2}z^2y^2\right]_{y=0}^{y=1-z} \\ &= \tfrac{1}{2}z^2(1-z)^2 \\ &= \tfrac{1}{2}z^2 - z^3 + \tfrac{1}{2}z^4. \end{aligned}$$

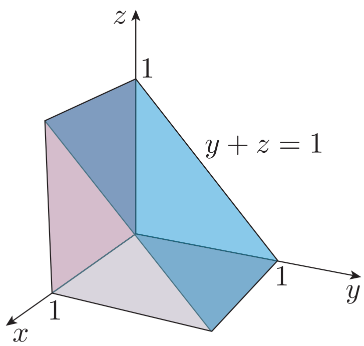
Finally, we integrate over z to obtain

$$\begin{aligned}\int_R f(x,y,z)\,dV &= \int_{z=0}^{z=1} \left(\tfrac{1}{2}z^2 - z^3 + \tfrac{1}{2}z^4\right) dz \\ &= \left[\tfrac{1}{6}z^3 - \tfrac{1}{4}z^4 + \tfrac{1}{10}z^5\right]_{z=0}^{z=1} \\ &= \tfrac{1}{6} - \tfrac{1}{4} + \tfrac{1}{10} = \tfrac{1}{60},\end{aligned}$$

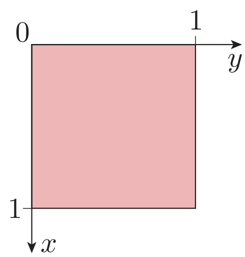
which does agree with Example 4.

Solution to Exercise 13

The region R and its projection onto the xy-plane are shown in the figure below.



(a)



(b)

The required volume integral is

$$\int_R f(x,y,z)\,dV = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1} \left(\int_{z=0}^{z=1-y} x^2yz\,dz \right) dy \right) dx.$$

Although the integrand is a product function, this volume integral cannot be expressed as the product of three ordinary integrals because the limits of integration are not all constants.

The integral over z gives

$$\begin{aligned}\int_{z=0}^{z=1-y} x^2yz\,dz &= \left[\tfrac{1}{2}x^2yz^2\right]_{z=0}^{z=1-y} \\ &= \tfrac{1}{2}x^2y(1-y)^2 = \tfrac{1}{2}x^2(y - 2y^2 + y^3).\end{aligned}$$

The integral over y then gives

$$\begin{aligned}\int_{y=0}^{y=1} \tfrac{1}{2}x^2(y - 2y^2 + y^3)\,dy &= \left[\tfrac{1}{2}x^2\left(\tfrac{1}{2}y^2 - \tfrac{2}{3}y^3 + \tfrac{1}{4}y^4\right)\right]_{y=0}^{y=1} \\ &= \tfrac{1}{24}x^2.\end{aligned}$$

Finally, integrating over x gives

$$\int_R f(x,y,z)\,dV = \int_{x=0}^{x=1} \tfrac{1}{24}x^2\,dx = \left[\tfrac{1}{72}x^3\right]_{x=0}^{x=1} = \tfrac{1}{72}.$$

Solution to Exercise 14

There is no need to draw diagrams in this case because the limits are given in the question. However, we need to ensure that the integrals are nested in the right way. The required volume is

$$V = \int_{z=0}^{z=1} \left(\int_{y=0}^{y=z} \left(\int_{x=0}^{x=y^2+z^2} 1\,dx \right) dy \right) dz.$$

The integral over x is chosen as the innermost integral because its limits depend on the other two variables of integration, y and z. The integral over y is done next because its limits depend on z. The integral over z is chosen as the outermost integral because its limits are constants.

Carrying out the integrals, we obtain

$$\begin{aligned} V &= \int_{z=0}^{z=1} \left(\int_{y=0}^{y=z} (y^2+z^2)\,dy \right) dz \\ &= \int_{z=0}^{z=1} \left[\tfrac{1}{3}y^3 + z^2 y\right]_{y=0}^{y=z} dz \\ &= \int_{z=0}^{z=1} \tfrac{4}{3}z^3\,dz \\ &= \left[\tfrac{1}{3}z^4\right]_{z=0}^{z=1} = \tfrac{1}{3} \simeq 0.33. \end{aligned}$$

Since the lengths are measured in metres, the volume is $0.33\,\mathrm{m}^3$.

Solution to Exercise 15

Recognising that $x^2 + y^2 = r^2$, the surface density function becomes

$$f(r,\phi) = \frac{C}{R^4}(2R^2 - r^2)$$

in polar coordinates. Note that we continue to use the symbol f for this function even though it is now expressed in terms of new variables. This convention was discussed at the start of this book.

The total number of bacteria on the dish is given by the area integral

$$\begin{aligned} N &= \int_{\phi=0}^{\phi=2\pi} \left(\int_{r=0}^{r=R} f(r,\phi)\,r\,dr \right) d\phi \\ &= \frac{C}{R^4} \int_{\phi=0}^{\phi=2\pi} \left(\int_{r=0}^{r=R} (2R^2 r - r^3)\,dr \right) d\phi. \end{aligned}$$

Carrying out the integral over r first gives

$$\begin{aligned} N &= \frac{C}{R^4} \int_{\phi=0}^{\phi=2\pi} \left[R^2 r^2 - \tfrac{1}{4}r^4\right]_{r=0}^{r=R} d\phi \\ &= \frac{C}{R^4} \int_{\phi=0}^{\phi=2\pi} \tfrac{3}{4}R^4\,d\phi. \end{aligned}$$

The final integral over ϕ is trivial, giving the answer

$$N = \frac{C}{R^4} \times 2\pi \times \tfrac{3}{4}R^4 = \tfrac{3}{2}\pi C.$$

Solution to Exercise 16

The semicircular region S is defined by $0 \le r \le R$ and $0 \le \phi \le \pi$. Recalling that $x = r\cos\phi$ and including the factor r required by the area element in polar coordinates, we obtain

$$\int_S x\,dA = \int_{\phi=0}^{\phi=\pi} \left(\int_{r=0}^{r=R} (r\cos\phi)\,r\,dr \right) d\phi.$$

The integrand is the product of a function of r and a function of ϕ, and the limits of integration are all constants. This allows us to write the area integral as a product of two ordinary integrals. Hence

$$\begin{aligned}\int_S x\,dA &= \int_{r=0}^{r=R} r^2\,dr \times \int_{\phi=0}^{\phi=\pi} \cos\phi\,d\phi \\ &= \tfrac{1}{3}R^3 \times \big[\sin\phi\big]_{\phi=0}^{\phi=\pi} = 0.\end{aligned}$$

This answer is not surprising: the region of integration is symmetrical about the y-axis, but the integrand is an odd function of x, so contributions from $x < 0$ cancel those from $x > 0$.

A similar calculation for $y = r\sin\phi$ gives

$$\begin{aligned}\int_S y\,dA &= \int_{r=0}^{r=R} r^2\,dr \times \int_{\phi=0}^{\phi=\pi} \sin\phi\,d\phi \\ &= \tfrac{1}{3}R^3 \times \big[-\cos\phi\big]_{\phi=0}^{\phi=\pi} = \tfrac{2}{3}R^3.\end{aligned}$$

Solution to Exercise 17

The area integral of e^{-r^2} over the entire xy-plane is

$$I = \int_{\phi=0}^{\phi=2\pi} \left(\int_{r=0}^{r=\infty} e^{-r^2}\,r\,dr \right) d\phi.$$

The integrand is a function of r only. This can be regarded as a product function where the function of ϕ is equal to 1. Also, the limits of integration are all constants. We can therefore write the area integral as the product of two ordinary integrals:

$$I = \int_{\phi=0}^{\phi=2\pi} 1\,d\phi \times \int_{r=0}^{r=\infty} e^{-r^2}\,r\,dr = 2\pi \int_{r=0}^{r=\infty} e^{-r^2}\,r\,dr.$$

This tactic works because r^2 in $\exp(-r^2)$ has a derivative that is proportional to the factor r in the integrand.

We make the substitution $u = r^2$. Then $du/dr = 2r$, and the new limits of integration are $u = 0$ and $u = \infty$. Hence

$$\begin{aligned}I &= 2\pi \int_{r=0}^{r=\infty} e^{-u}\,\tfrac{1}{2}\frac{du}{dr}\,dr \\ &= \pi \int_{u=0}^{u=\infty} e^{-u}\,du \\ &= \pi\left[-e^{-u}\right]_{u=0}^{u=\infty} = \pi\left(-e^{-\infty} + e^0\right) = \pi,\end{aligned}$$

where we have interpreted $e^{-\infty}$ as being equal to zero. This is appropriate because e^{-x} tends to zero as x tends to infinity.

Solution to Exercise 18

The region of integration corresponds to $2 \leq r \leq 5$, $0 \leq \phi \leq 2\pi$ and $-1 \leq z \leq 1$, so the required volume integral of rz^2 is

$$I = \int_{z=-1}^{z=1} \left(\int_{\phi=0}^{\phi=2\pi} \left(\int_{r=2}^{r=5} r^2 z^2 \, dr \right) d\phi \right) dz,$$

where the extra factor of r in the integrand comes from the volume element in cylindrical coordinates.

In an integral such as this, where the integrand is a product of a function of r and a function of z, and the limits of integration are all constants, we can write the integral as a product of three ordinary integrals:

$$\begin{aligned} I &= \int_{z=-1}^{z=1} z^2 \, dz \times \int_{\phi=0}^{\phi=2\pi} 1 \, d\phi \times \int_{r=2}^{r=5} r^2 \, dr \\ &= \left[\tfrac{1}{3} z^3\right]_{z=-1}^{z=1} \times 2\pi \times \left[\tfrac{1}{3} r^3\right]_{r=2}^{r=5} \\ &= \tfrac{2}{3} \times 2\pi \times \tfrac{117}{3} \\ &= 52\pi. \end{aligned}$$

Solution to Exercise 19

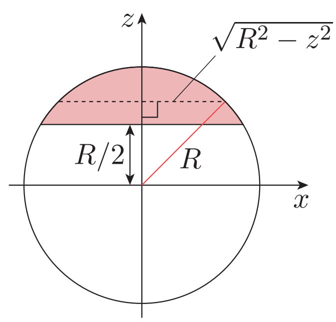
Using equation (27), the volume of the rugby ball is

$$\begin{aligned} V &= \pi \int_{z=-b}^{z=b} a^2 \left(1 - \frac{z^2}{b^2}\right) dz \\ &= \pi a^2 \left[z - \frac{z^3}{3b^2} \right]_{z=-b}^{z=b} \\ &= \tfrac{4}{3} \pi a^2 b. \end{aligned}$$

Check: When $a = b = R$, the rugby ball becomes a sphere of radius R, and the volume becomes $4\pi R^3/3$, as expected.

Solution to Exercise 20

The figure in the margin shows a cross-section through the spherical cap (shaded). At a given value of z, the surface of the cap has cylindrical radial coordinate



$$r = r_{\max}(z) = \sqrt{R^2 - z^2}.$$

The spherical cap has $R/2 \leq z \leq R$, so its volume is

$$\begin{aligned} V &= \pi \int_{z=R/2}^{z=R} \left(R^2 - z^2\right) dz \\ &= \pi \left[R^2 z - \tfrac{1}{3} z^3\right]_{z=R/2}^{z=R} \\ &= \pi R^3 \left(1 - \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{24}\right) \\ &= \tfrac{5}{24} \pi R^3. \end{aligned}$$

Solution to Exercise 21

The required volume integral is

$$I = \int_{\phi=0}^{\phi=2\pi} \left(\int_{\theta=0}^{\theta=\pi} \left(\int_{r=1}^{r=2} \frac{\sin\theta}{r}\, r^2 \sin\theta \, dr \right) d\theta \right) d\phi.$$

The integrand is a product function, and the limits of integration are all constants, so the volume integral can be written as

$$I = \int_{\phi=0}^{\phi=2\pi} 1\, d\phi \times \int_{\theta=0}^{\theta=\pi} \sin^2\theta \, d\theta \times \int_{r=1}^{r=2} r\, dr.$$

Using the trigonometric identity $\sin^2\theta = \frac{1}{2}\big(1-\cos(2\theta)\big)$, we get

$$\int_{\theta=0}^{\theta=\pi} \sin^2\theta\, d\theta = \tfrac{1}{2}\int_{\theta=0}^{\theta=\pi} \big(1-\cos(2\theta)\big)\, d\theta = \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin(2\theta)\right]_{\theta=0}^{\theta=\pi} = \tfrac{1}{2}\pi.$$

So the required volume integral is

$$I = 2\pi \times \tfrac{1}{2}\pi \times \left[\tfrac{1}{2}r^2\right]_{r=1}^{r=2} = \tfrac{3}{2}\pi^2.$$

Solution to Exercise 22

(a) The variable r is the radial coordinate of spherical coordinates, so the volume integral is

$$\int_{\text{sphere}} f\, dV = \int_{\phi=0}^{\phi=2\pi} \left(\int_{\theta=0}^{\theta=\pi} \left(\int_{r=0}^{r=R} f(r)\, r^2 \sin\theta\, dr \right) d\theta \right) d\phi.$$

The integrand is a product function, and the limits of integration are all constants, so the volume integral can be split into the product of three ordinary integrals:

$$\begin{aligned} \int_{\text{sphere}} f\, dV &= \int_{\phi=0}^{\phi=2\pi} 1\, d\phi \times \int_{\theta=0}^{\theta=\pi} \sin\theta\, d\theta \times \int_0^R f(r)\, r^2\, dr \\ &= 2\pi \times \big[-\cos\theta\big]_{\theta=0}^{\theta=\pi} \times \int_0^R f(r)\, r^2\, dr \\ &= 4\pi \int_0^R f(r)\, r^2\, dr, \end{aligned}$$

as required.

(b) The quantity $\sqrt{x^2+y^2+z^2}$ is the distance of a point from the origin, which is equal to the radial coordinate r in spherical coordinates. This can be established more formally using equations (28), which give

$$\begin{aligned} x^2+y^2+z^2 &= r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta \\ &= r^2\sin^2\theta\,(\cos^2\phi+\sin^2\phi) + r^2\cos^2\theta \\ &= r^2(\sin^2\theta+\cos^2\theta) = r^2. \end{aligned}$$

Hence the result of part (a) gives

$$\int_{\text{sphere}} \sqrt{x^2+y^2+z^2}\, dV = 4\pi \int_0^R r^3\, dr = \pi R^4.$$

Solution to Exercise 23

Using equation (33), we get the following results. In cylindrical coordinates,

$$\delta V = 1 \times r \times 1\,\delta r\,\delta\phi\,\delta z = r\,\delta r\,\delta\phi\,\delta z,$$

and in spherical coordinates,

$$\delta V = 1 \times r \times r\sin\theta\,\delta r\,\delta\theta\,\delta\phi = r^2\sin\theta\,\delta r\,\delta\theta\,\delta\phi.$$

Both of these expressions agree with our previous results.

Solution to Exercise 24

Taking partial derivatives of x, y and z with respect to r, θ and ϕ, we have

$$\begin{aligned}
&\frac{\partial x}{\partial r} = \sin\theta\cos\phi, \quad \frac{\partial y}{\partial r} = \sin\theta\sin\phi, \quad \frac{\partial z}{\partial r} = \cos\theta,\\
&\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, \quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta,\\
&\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi, \quad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi, \quad \frac{\partial z}{\partial \phi} = 0.
\end{aligned}$$

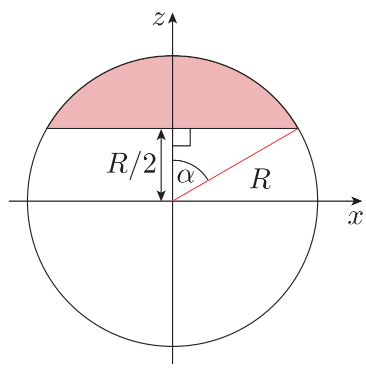
Hence the scale factors are

$$\begin{aligned}
h_r &= \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}\\
&= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta}\\
&= 1,\\
h_\theta &= \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}\\
&= \sqrt{r^2(\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta)}\\
&= r,\\
h_\phi &= \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi + 0}\\
&= \sqrt{r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)}\\
&= r\sin\theta,
\end{aligned}$$

where the square roots have been taken in the knowledge that $r \geq 0$ and $\sin\theta \geq 0$ for spherical coordinates. Our answers agree with equation (36).

Solution to Exercise 25

The cross-section in the figure shows that the spherical cap (shaded) extends from $\theta = 0$ to $\theta = \alpha$, where $\cos\alpha = (\frac{1}{2}R)/R = \frac{1}{2}$.



Using the area element in equation (40), we get

$$\begin{aligned}
\text{area of cap} &= \int_{\phi=0}^{\phi=2\pi}\left(\int_{\theta=0}^{\theta=\alpha} R^2\sin\theta\,d\theta\right)d\phi\\
&= R^2\int_{\phi=0}^{\phi=2\pi}\Big[-\cos\theta\Big]_{\theta=0}^{\theta=\alpha}\,d\phi\\
&= 2\pi R^2(1-\cos\alpha).
\end{aligned}$$

In the present case, $\cos\alpha = 1/2$, so the area of the cap is πR^2. For the (practically) spherical surface of the Earth, this means that there is the same amount of area north of latitude 30° (the latitude of Cairo) as there is between latitude 30° and the Equator.

Note: This calculation works very well in spherical coordinates. By contrast, finding the *volume* of a spherical cap works better in cylindrical coordinates (see Exercise 20). In the volume integral the flat base of the spherical cap must be taken into account, and this is more simply described in cylindrical, rather than spherical, coordinates.

Solution to Exercise 26

Expanding the determinant, we obtain

$$\begin{aligned}\mathbf{J} &= R^2\sin^2\theta\cos\phi\,\mathbf{i} + R^2\sin^2\theta\sin\phi\,\mathbf{j} + R^2\cos\theta\sin\theta(\cos^2\phi + \sin^2\phi)\,\mathbf{k}\\ &= R^2\sin^2\theta\cos\phi\,\mathbf{i} + R^2\sin^2\theta\sin\phi\,\mathbf{j} + R^2\cos\theta\sin\theta\,\mathbf{k}\\ &= R^2\sin\theta(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}).\end{aligned}$$

The square of the magnitude of this vector is

$$\begin{aligned}|\mathbf{J}|^2 &= R^4\sin^2\theta\left(\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta\right)\\ &= R^4\sin^2\theta\left(\sin^2\theta + \cos^2\theta\right)\\ &= R^4\sin^2\theta.\end{aligned}$$

We have $R > 0$ and $0 \le \theta \le \pi$, so $R^2\sin\theta \ge 0$. Hence $|\mathbf{J}| = R^2\sin\theta$, and the area of a surface element is

$$\delta A = |\mathbf{J}|\,\delta\theta\,\delta\phi = R^2\sin\theta\,\delta\theta\,\delta\phi.$$

Solution to Exercise 27

Differentiating the functions for x, y and z with respect to r and ϕ, we get

$$\begin{aligned}&\frac{\partial x}{\partial r} = \cos\phi, \quad \frac{\partial y}{\partial r} = \sin\phi, \quad \frac{\partial z}{\partial r} = \frac{r}{R},\\ &\frac{\partial x}{\partial \phi} = -r\sin\phi, \quad \frac{\partial y}{\partial \phi} = r\cos\phi, \quad \frac{\partial z}{\partial \phi} = 0,\end{aligned}$$

so the Jacobian vector is

$$\begin{aligned}\mathbf{J} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\phi & \sin\phi & r/R\\ -r\sin\phi & r\cos\phi & 0\end{vmatrix}\\ &= -\frac{r^2}{R}\cos\phi\,\mathbf{i} - \frac{r^2}{R}\sin\phi\,\mathbf{j} + r(\cos^2\phi + \sin^2\phi)\,\mathbf{k}\\ &= -\frac{r^2}{R}\cos\phi\,\mathbf{i} - \frac{r^2}{R}\sin\phi\,\mathbf{j} + r\,\mathbf{k}.\end{aligned}$$

Hence

$$|\mathbf{J}|^2 = \frac{r^4}{R^2}(\cos^2\phi + \sin^2\phi) + r^2 = r^2\left(1 + \frac{r^2}{R^2}\right).$$

The area element is therefore

$$\delta A = |\mathbf{J}|\,\delta r\,\delta\phi = \left(1 + \frac{r^2}{R^2}\right)^{1/2} r\,\delta r\,\delta\phi,$$

and the surface area of the dish is

$$\text{surface area} = \int_{\phi=0}^{\phi=2\pi} \left(\int_{r=0}^{r=R} \left(1 + \frac{r^2}{R^2}\right)^{1/2} r\,dr \right) d\phi.$$

To evaluate the integral over r, we make the substitution $u = 1 + r^2/R^2$. Then $du/dr = 2r/R^2$, and the lower and upper limits of integration become $u = 1$ and $u = 2$. So we get

$$\begin{aligned}
\text{surface area} &= \int_{\phi=0}^{\phi=2\pi} \left(\int_{r=0}^{r=R} u^{1/2} \times \tfrac{1}{2}R^2 \frac{du}{dr}\,dr \right) d\phi \\
&= \tfrac{1}{2}R^2 \int_{\phi=0}^{\phi=2\pi} \left(\int_{u=1}^{u=2} u^{1/2}\,du \right) d\phi \\
&= \tfrac{1}{2}R^2 \int_{\phi=0}^{\phi=2\pi} \left[\tfrac{2}{3}u^{3/2}\right]_{u=1}^{u=2} d\phi \\
&= \tfrac{1}{2}R^2\,\tfrac{2}{3} \int_{\phi=0}^{\phi=2\pi} (\sqrt{8} - 1)\,d\phi = \tfrac{2}{3}\pi(\sqrt{8} - 1)R^2.
\end{aligned}$$

Solution to Exercise 28

Taking partial derivatives of x, y and z with respect to u and v, we get

$$\begin{aligned}
&\frac{\partial x}{\partial u} = u, \quad \frac{\partial y}{\partial u} = u, \quad \frac{\partial z}{\partial u} = v, \\
&\frac{\partial x}{\partial v} = v, \quad \frac{\partial y}{\partial v} = -v, \quad \frac{\partial z}{\partial v} = u.
\end{aligned}$$

Hence

$$\mathbf{J} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u & u & v \\ v & -v & u \end{vmatrix} = (u^2 + v^2)\,\mathbf{i} - (u^2 - v^2)\,\mathbf{j} - 2uv\,\mathbf{k},$$

and

$$\begin{aligned}
|\mathbf{J}|^2 &= (u^4 + 2u^2v^2 + v^4) + (u^4 - 2u^2v^2 + v^4) + 4u^2v^2 \\
&= 2(u^4 + 2u^2v^2 + v^4) \\
&= 2(u^2 + v^2)^2.
\end{aligned}$$

The required surface area is

$$\begin{aligned}
\text{surface area} &= \int_{v=0}^{v=1} \left(\int_{u=0}^{u=1} \sqrt{2}(u^2 + v^2)\,du \right) dv \\
&= \sqrt{2} \int_{v=0}^{v=1} \left[\tfrac{1}{3}u^3 + uv^2\right]_{u=0}^{u=1} dv \\
&= \sqrt{2} \int_{v=0}^{v=1} \left(\tfrac{1}{3} + v^2\right) dv = \sqrt{2}\left[\tfrac{1}{3}v + \tfrac{1}{3}v^3\right]_{v=0}^{v=1} = \tfrac{2\sqrt{2}}{3}.
\end{aligned}$$

Acknowledgements

Grateful acknowledgement is made to the following source:

Figure 4: NASA.

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